

# CUP PRODUCTS, THE HEISENBERG GROUP, AND CODIMENSION TWO ALGEBRAIC CYCLES

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**ABSTRACT.** We define higher categorical invariants (gerbes) of codimension two algebraic cycles and provide a categorical interpretation of the intersection of divisors on a smooth proper algebraic variety. This generalization of the classical relation between divisors and line bundles furnishes a new perspective on the Bloch-Quillen formula.

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*Des buissons lumineux fusaient comme des gerbes;  
 Mille insectes, tels des prismes, vibraient dans l'air;  
 Le vent jouait avec l'ombre des lilas clairs,  
 Sur le tissu des eaux et les nappes de l'herbe.  
 Un lion se couchait sous des branches en fleurs;  
 Le daim flexible errait là-bas, près des panthères;  
 Et les paons déployaient des faisceaux de lueurs  
 Parmi les phlox en feu et les lys de lumière.*  
 –Emile Verhaeren (1855-1916),  
*Le paradis* (Les rythmes souverains)

## 1. INTRODUCTION

This aim of this paper is to define higher categorical invariants (gerbes) of codimension two algebraic cycles and provide a categorical interpretation of the intersection of divisors on a smooth proper algebraic variety. This generalization of the classical relation between divisors and line bundles furnishes a new perspective on the classical Bloch-Quillen formula relating Chow groups and algebraic K-theory.

Our work is motivated by the following three basic questions.

- (i) Let  $A$  and  $B$  be abelian sheaves on a manifold (or algebraic variety)  $X$ . Given  $\alpha \in H^1(X, A)$  and  $\beta \in H^1(X, B)$ , one has their cup-product  $\alpha \cup \beta \in H^2(X, A \otimes B)$ . We recall that  $H^1$  and  $H^2$  classify equivalence classes of torsors and gerbes<sup>1</sup>:

$$\begin{aligned} H^1(X, A) &\longleftrightarrow \text{Isomorphism classes of } A\text{-torsors} \\ H^2(X, A) &\longleftrightarrow \text{Isomorphism classes of } A\text{-gerbes;} \end{aligned}$$

we may pick torsors  $P$  and  $Q$  representing  $\alpha$  and  $\beta$  and ask

**Question 1.1.** *Given  $P$  and  $Q$ , is there a natural construction of a gerbe  $G_{P,Q}$  which manifests the cohomology class  $\alpha \cup \beta = [P] \cup [Q]$ ?*

The above question admits the following algebraic-geometric analogue.

- (ii) Let  $X$  be a smooth proper variety over a field  $F$ . Let  $Z^i(X)$  be the abelian group of algebraic cycles of codimension  $i$  on  $X$  and let  $CH^i(X)$  be the Chow group of algebraic cycles of codimension  $i$  modulo rational equivalence. The isomorphism

$$CH^1(X) \xrightarrow{\sim} H^1(X, \mathcal{O}^*)$$

connects (Weil) divisors and invertible sheaves (or  $\mathbb{G}_m$ -torsors). While divisors form a group,  $\mathbb{G}_m$ -torsors on  $X$  form a Picard category  $\text{TORS}_X(\mathbb{G}_m)$  with the monoidal structure provided by the Baer sum of torsors. Any divisor  $D$  determines a  $\mathbb{G}_m$ -torsor  $\mathcal{O}_D$ ; the torsor  $\mathcal{O}_{D+D'}$  is isomorphic to the Baer sum of  $\mathcal{O}_D$  and  $\mathcal{O}_{D'}$ . In other words, one has an additive map [20, II, Proposition 6.13]

$$(1.0.1) \quad Z^1(X) \rightarrow \text{TORS}_X(\mathbb{G}_m) \quad D \mapsto \mathcal{O}_D.$$

**Question 1.2.** *What is a natural generalization of (1.0.1) to higher codimension cycles?*

Since  $\text{TORS}_X(\mathbb{G}_m)$  is a Picard category, one could expect the putative additive maps on  $Z^i(X)$  to land in Picard categories or their generalizations.

**Question 1.3.** *Is there a categorification of the intersection pairing*

$$(1.0.2) \quad CH^1(X) \times CH^1(X) \rightarrow CH^2(X)?$$

More generally, one can ask for a categorical interpretation of the entire Chow ring of  $X$ .

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<sup>1</sup>For us, the term "gerbe" signifies a stack in groupoids which is locally non-empty and locally connected (§2.1). It is slightly different from the ancient gerbes of *Acids, alkalies and salts: their manufacture and applications, Volume 2* (1865) by Thomas Richardson and Henry Watts, pp. 567-569:

§4. Gerbes

This firework is made in various ways, generally throwing up a luminous and sparkling jet of fire, somewhat resembling a water-spout: hence its name. Gerbes consist of a straight, cylindrical case, sometime made with wrought iron (if the gerbe is of large dimensions). ... Mr. Darby has invented an entirely novel and beautiful gerbe, called the Italian gerbe..."

**Main results.** Our first result is an affirmative answer to Question 1.1; the key observation is that a certain Heisenberg group *animates* the cup-product.

**Theorem 1.4.** *Let  $A, B$  be abelian sheaves on a topological space or scheme  $X$ .*

- (i) *There is a canonical functorial Heisenberg<sup>2</sup> sheaf  $H_{A,B}$  on  $X$  which sits in an exact sequence*

$$0 \rightarrow A \otimes B \rightarrow H_{A,B} \rightarrow A \times B \rightarrow 0;$$

*the sheaf  $H_{A,B}$  (of non-abelian groups) is a central extension of  $A \times B$  by  $A \otimes B$ .*

- (ii) *The associated boundary map*

$$\partial : H^1(X, A) \times H^1(X, B) = H^1(X, A \times B) \rightarrow H^2(X, A \otimes B)$$

*sends the class  $(\gamma, \delta)$  to the cup-product  $\gamma \cup \delta$ .*

- (iii) *Given torsors  $P$  and  $Q$  for  $A$  and  $B$ , view  $P \times Q$  as a  $A \times B$ -torsor on  $X$ . Let  $\mathcal{G}_{P,Q}$  be the gerbe of local liftings (see §2.2) of  $P \times Q$  to a  $H_{A,B}$ -torsor; its band is  $A \otimes B$  and its class in  $H^2(X, A \otimes B)$  is  $[P] \cup [Q]$ .*
- (iv) *The gerbe  $\mathcal{G}_{P,Q}$  is covariant functorial in  $A$  and  $B$  and contravariant functorial in  $X$ .*
- (v) *The gerbe  $\mathcal{G}_{P,Q}$  is trivial (equivalent to the stack of  $A \otimes B$ -torsors) if either  $P$  or  $Q$  is trivial.*

We prove this theorem over a general site  $\mathcal{C}$ . We also provide a natural interpretation of the (class of the) Heisenberg sheaf in terms of maps of Eilenberg-Mac Lane objects in §3.4; it is astonishing that the explicit cocycle (3.1.3) for the Heisenberg group (when  $X = \text{a point}$ ) turns out to coincide with the map on the level of Eilenberg-Mac Lane objects over a general site  $\mathcal{C}$ ; cf. 3.4.

Here is another rephrasing of Theorem 1.4: For abelian sheaves  $A$  and  $B$  on a site  $\mathcal{C}$ , there is a natural bimonoidal functor

$$(1.0.3) \quad \text{TORS}_{\mathcal{C}}(A) \times \text{TORS}_{\mathcal{C}}(B) \longrightarrow \text{GERBES}_{\mathcal{C}}(A \otimes B) \quad (P, Q) \mapsto \mathcal{G}_{P,Q}$$

where  $\text{TORS}_{\mathcal{C}}(A)$ ,  $\text{TORS}_{\mathcal{C}}(B)$  are the Picard categories of  $A$  and  $B$ -torsors on  $\mathcal{C}$  and  $\text{GERBES}_{\mathcal{C}}(A \otimes B)$  is the Picard 2-category of  $A \otimes B$ -gerbes on  $\mathcal{C}$ . Thus, Theorem 1.4 constitutes a categorification of the cup-product map

$$(1.0.4) \quad \cup : H^1(A) \times H^1(B) \rightarrow H^2(A \otimes B).$$

Let us turn to Questions 1.2 and 1.3. Suppose that  $D$  and  $D'$  are divisors on  $X$  which intersect in the codimension-two cycle  $D.D'$ . Applying Theorem 1.4 to  $\mathcal{O}_D$  and  $\mathcal{O}_{D'}$  with  $A = B = \mathbb{G}_m$ , one has a  $\mathbb{G}_m \otimes \mathbb{G}_m$ -gerbe  $\mathcal{G}_{D,D'}$  on  $X$ . We now invoke the isomorphisms (the second is the fundamental Bloch-Quillen isomorphism)

$$\mathbb{G}_m \xrightarrow{\sim} \mathcal{K}_1, \quad CH^i(X) \xrightarrow[(5.1.3)]{\sim} H^i(X, \mathcal{K}_i)$$

where  $\mathcal{K}_i$  is the Zariski sheaf associated with the presheaf  $U \mapsto K_i(U)$ .

Pushforward of  $\mathcal{G}_{D,D'}$  along  $\mathcal{K}_1 \times \mathcal{K}_1 \rightarrow \mathcal{K}_2$  gives a  $\mathcal{K}_2$ -gerbe still denoted  $\mathcal{G}_{D,D'}$ ; we call this the Heisenberg gerbe attached to the codimension-two cycle  $D.D'$ . This raises the possibility of relating  $\mathcal{K}_2$ -gerbes and codimension-two cycles on  $X$ , implicit in (5.1.3).

**Theorem 1.5.** (i) *Any codimension-two cycle  $\alpha \in Z^2(X)$  determines a  $\mathcal{K}_2$ -gerbe  $\mathcal{C}_\alpha$  on  $X$ .*

(ii) *the class of  $\mathcal{C}_\alpha$  in  $H^2(X, \mathcal{K}_2)$  corresponds to  $\alpha \in CH^2(X)$  under the Bloch-Quillen map (5.1.3).*

(iii) *the gerbe  $\mathcal{C}_{\alpha+\alpha'}$  is equivalent to the Baer sum of  $\mathcal{C}_\alpha$  and  $\mathcal{C}_{\alpha'}$ .*

(iv)  *$\mathcal{C}_\alpha$  and  $\mathcal{C}_{\alpha'}$  are equivalent as  $\mathcal{K}_2$ -gerbes if and only if  $\alpha = \alpha'$  in  $CH^2(X)$ .*

The Gersten gerbe  $\mathcal{C}_\alpha$  of  $\alpha$  admits a geometric description, closely analogous to that of the  $\mathbb{G}_m$ -torsor  $\mathcal{O}_D$  of a divisor  $D$ ; see Remark 5.6. The Gersten sequence (5.1.1) is key to the construction of  $\mathcal{C}_\alpha$ . One has an additive map

$$(1.0.5) \quad Z^2(X) \rightarrow \text{GERBES}_X(\mathcal{K}_2) \quad \alpha \mapsto \mathcal{C}_\alpha.$$

When  $\alpha = D.D'$  is the intersection of two divisors, there are two  $\mathcal{K}_2$ -gerbes attached to it: the Heisenberg gerbe  $\mathcal{G}_{D,D'}$  and the Gersten gerbe  $\mathcal{C}_\alpha$ ; these are abstractly equivalent as their classes in  $H^2(X, \mathcal{K}_2)$  correspond to  $\alpha$ . More is possible.

<sup>2</sup>The usual Heisenberg group, a central extension of  $A \times B$  by  $\mathbb{C}^*$ , arises from a biadditive map  $A \times B \rightarrow \mathbb{C}^*$ .

**Theorem 1.6.** *If  $\alpha \in Z^2(X)$  is the intersection  $D.D'$  of divisors  $D, D' \in Z^1(X)$ , then there is a natural equivalence  $\Theta : \mathcal{C}_\alpha \rightarrow \mathcal{G}_{D,D'}$  between the Gersten and Heisenberg  $\mathcal{K}_2$ -gerbes attached to  $\alpha = D.D'$ .*

Thus, Theorems 1.4, 1.5, 1.6 together provide the following commutative diagram thereby answering Question 1.3:

$$\begin{array}{ccc}
Z^1(X) \times Z^1(X) & \xrightarrow{\text{no map}} & Z^2(X) \\
\downarrow (1.0.1) & & \downarrow (1.0.5) \\
\text{TORS}_X(\mathbb{G}_m) \times \text{TORS}_X(\mathbb{G}_m) & \xrightarrow{(1.0.3)} & \text{GERBES}_X(\mathcal{K}_2) \\
\downarrow & & \downarrow \\
CH^1(X) \times CH^1(X) & \xrightarrow{(1.0.2)} & CH^2(X).
\end{array}$$

We begin with a review of the basic notions and tools (lifting gerbe, four-term complexes) in §2 and then present the construction and properties of the Heisenberg group in §3 before proving Theorem 1.4. After a quick discussion of various examples in §4, we turn to codimension-two algebraic cycles in §5 and construct the Gersten gerbe  $\mathcal{C}_\alpha$  and prove Theorems 1.5, 1.6 using the tools in §2.

**Dictionary for codimension two cycles.** The above results indicate the viability of viewing  $\mathcal{K}_2$ -gerbes as natural invariants of codimension-two cycles on  $X$ . Additional evidence is given by the following points:

- $\mathcal{K}_2$ -gerbes are present (albeit implicitly) in the Bloch-Quillen formula (5.1.3) for  $i = 2$ .
- The Picard category  $\mathfrak{P} = \text{TORS}_X(\mathbb{G}_m)$  of  $\mathbb{G}_m$ -torsors on  $X$  satisfies

$$\pi_1(\mathfrak{P}) = H^0(X, \mathcal{O}^*) = CH^1(X, 1), \quad \pi_0(\mathfrak{P}) = H^1(X, \mathcal{O}^*) = CH^1(X).$$

Similarly, the Picard 2-category  $\mathfrak{E} = \text{GERBES}_X(\mathcal{K}_2)$  of  $\mathcal{K}_2$ -gerbes is closely related to Bloch's higher Chow complex [3] in codimension two:

$$\pi_2(\mathfrak{E}) = H^0(X, \mathcal{K}_2) = CH^2(X, 2), \quad \pi_1(\mathfrak{E}) = H^1(X, \mathcal{K}_2) = CH^2(X, 1), \quad \pi_0(\mathfrak{E}) = H^2(X, \mathcal{K}_2) \stackrel{(5.1.3)}{=} CH^2(X).$$

- The additive map arising from Theorem 1.5

$$Z^2(X) \rightarrow \text{GERBES}_X(\mathcal{K}_2), \quad \alpha \mapsto \mathcal{C}_\alpha$$

gives the Bloch-Quillen isomorphism (5.1.3) on the level of  $\pi_0$ . It provides an answer to Question 1.2 for codimension two cycles.

- The Gersten gerbe  $\mathcal{C}_\alpha$  admits a simple algebro-geometric description (Remark 5.5): Any  $\alpha$  determines a  $K_2^\eta/\mathcal{K}_2$ -torsor; then  $\mathcal{C}_\alpha$  is the gerbe of liftings of this torsor to a  $K_2^\eta$ -torsor on  $X$ .
- The gerbe  $\mathcal{C}_\alpha$  is canonically trivial outside of the support of  $\alpha$  (Remark 5.5).
- Pushing the Gersten gerbe  $\mathcal{C}_\alpha$  along the map  $\mathcal{K}_2 \rightarrow \Omega^2$  produces an  $\Omega^2$ -gerbe which manifests the (de Rham) cycle class of  $\alpha$  in  $H^2(X, \Omega^2)$ .

The map (1.0.1) is a part of the marvellous dictionary [20, II, §6] arising from the divisor sequence (5.2.1):

$$\text{Divisors} \longleftrightarrow \text{Cartier divisors} \longleftrightarrow \mathcal{K}_1\text{-torsors} \longleftrightarrow \text{Line bundles} \longleftrightarrow \text{Invertible sheaves}.$$

More generally, from the Gersten sequence (5.1.1) we obtain the following:

$$\begin{aligned}
Z^1(X) &\xrightarrow{\cong} H^0(X, K_1^\eta/\mathcal{K}_1) \twoheadrightarrow H^1(X, \mathcal{K}_1) \cong CH^1(X) \\
Z^2(X) &\twoheadrightarrow H^1(X, K_2^\eta/\mathcal{K}_2) \xrightarrow{\cong} H^2(X, \mathcal{K}_2) \cong CH^2(X).
\end{aligned}$$

Inspired by this and by ref. [2, Definition 3.2], we call  $K_2^\eta/\mathcal{K}_2$ -torsors as *codimension-two Cartier cycles* on  $X$ . Thus the analog for codimension two cycles of the above dictionary reads

$$\text{Codimension two cycles} \longleftrightarrow \text{Cartier cycles} \longleftrightarrow \mathcal{K}_2\text{-gerbes}.$$

Since the Gersten sequence (5.1.1) exists for all  $\mathcal{K}_i$ , it is possible to generalize Theorem 1.5 to higher codimensions thereby answering Question 1.2; however, this involves *higher gerbes*. Any cycle of codimension  $i > 2$  determines a higher gerbe [6] with band  $\mathcal{K}_i$  (see §5.7 for an example); this provides a new perspective

<sup>3</sup>Let  $\eta : \text{Spec } F_X \rightarrow X$  be the generic point of  $X$  and write  $K_i^\eta$  for the sheaf  $\eta_* K_i(F_X)$ ; one has the map  $\mathcal{K}_i \rightarrow K_i^\eta$ .

on the Bloch-Quillen formula (5.1.3). The higher dimensional analogues of (1.0.3), (1.0.2), and Theorem 1.5 will be pursued elsewhere.

Other than the classical Hartshorne-Serre correspondence between certain codimension-two cycles and certain rank two vector bundles, we are not aware of any generalizations of this dictionary to higher codimension. In particular, our idea of attaching *a higher-categorical invariant to a higher codimension cycle* seems new in the literature. We expect that Picard  $n$ -categories play a role in the functorial Riemann-Roch program of Deligne [14].

Our results are related to and inspired by the beautiful work of S. Bloch [2], L. Breen [7], J.-L. Brylinski [9], A. N. Parshin [27], B. Poonen - E. Rains [28], and D. Ramakrishnan [30] (see §4). Brylinski's hope<sup>4</sup> [9, Introduction] for a higher-categorical geometrical interpretation of the regulator maps from algebraic K-theory to Deligne cohomology was a major catalyst. In a forthcoming paper, we will investigate the relations between the Gersten gerbe and Deligne cohomology.

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**Notations and conventions.** Let  $\mathcal{C}$  be a site. We write  $\mathcal{C}^\sim$  for the topos of sheaves over  $\mathcal{C}$ ,  $\mathcal{C}_{\text{ab}}^\sim$  the abelian group objects of  $\mathcal{C}^\sim$ , namely the abelian sheaves on  $\mathcal{C}$ , and by  $\mathcal{C}_{\text{grp}}^\sim$  the sheaves of groups on  $\mathcal{C}$ . Our notation for cohomology is as follows. For an abelian object  $A$  of a topos  $\mathcal{T}$ ,  $H^i(A)$  denotes the cohomology of the terminal object  $e \in \mathcal{T}$  with coefficients in  $A$ , namely  $i^{\text{th}}$  derived functor of  $\text{Hom}_{\mathcal{T}}(e, A)$ . This is the same as  $\text{Ext}_{\mathcal{T}_{\text{ab}}}^i(\mathbb{Z}, A)$ . More generally,  $H^i(X, A)$  denotes the cohomology of  $A$  in the topos  $\mathcal{T}/X$ . We use  $\mathbf{H}$  for hypercohomology.

## 2. PRELIMINARIES

**2.1. Abelian Gerbes** [18, 15, 6]. A gerbe  $\mathcal{G}$  over a site  $\mathcal{C}$  is a *stack in groupoids which is locally non-empty and locally connected*.

$\mathcal{G}$  is locally nonempty if for every object  $U$  of  $\mathcal{C}$  there is a cover, say a local epimorphism,  $V \rightarrow U$  such that the category  $\mathcal{G}(V)$  is nonempty; it is locally connected if given objects  $x, y \in \mathcal{G}(U)$  as above, then, locally on  $U$ , the sheaf  $\text{Hom}(x, y)$  defined above has sections. For each object  $x$  over  $U$  we can introduce the automorphism sheaf  $\text{Aut}_{\mathcal{G}}(x)$ , and by local connectedness all these automorphism sheaves are (non canonically) isomorphic.

In the sequel we will only work with *abelian* gerbes, where there is a coherent identification between the automorphism sheaves  $\text{Aut}_{\mathcal{G}}(x)$ , for any choice of an object  $x$  of  $\mathcal{G}$ , and a fixed sheaf of groups  $G$ . In this case  $G$  is necessarily abelian<sup>5</sup>, and the class of  $\mathcal{G}$  determines an element in  $H^2(G)$ , [6, §2] (and also [23]), where  $H^i(G) = \text{Ext}_{\mathcal{C}_{\text{ab}}^\sim}^i(\mathbb{Z}, G)$  denotes the standard cohomology with coefficients in the abelian sheaf  $G$  in the topos  $\mathcal{C}^\sim$  of sheaves over  $\mathcal{C}$ .

Let us briefly recall how the class of  $\mathcal{G}$  is obtained using a Čech type argument. Assume for simplicity that the site  $\mathcal{C}$  has pullbacks. Let  $\mathcal{U} = \{U_i\}$  be a cover of an object  $X$  of  $\mathcal{C}$ . Let  $x_i$  be a choice of an object of  $\mathcal{G}(U_i)$ . For simplicity, let us assume that we can find morphisms  $\alpha_{ij}: x_j|_{U_{ij}} \rightarrow x_i|_{U_{ij}}$ . The class of  $\mathcal{G}$  will be represented by the 2-cocycle  $\{c_{ijk}\}$  of  $\mathcal{U}$  with values in  $G$  obtained in the standard way as the deviation for  $\{\alpha_{ij}\}$  from satisfying the cocycle condition:

$$\alpha_{ij} \circ \alpha_{jk} = c_{ijk} \circ \alpha_{ik}.$$

In the above identity—which defines it— $c_{ijk} \in \text{Aut}(x_i|_{U_{ijk}}) \cong G|_{U_{ijk}}$ . It is obvious that  $\{c_{ijk}\}$  is a cocycle.

Returning to stacks for a moment, a stack  $\mathcal{G}$  determines an object  $\pi_0(\mathcal{G})$ , defined as the sheaf associated to the presheaf of connected components of  $\mathcal{G}$ , where the latter is the presheaf that to each object  $U$  of  $\mathcal{C}$  assigns the set of isomorphism classes of objects of  $\mathcal{G}(U)$ . By definition, if  $\mathcal{G}$  is a gerbe, then  $\pi_0(\mathcal{G}) = *$ . In general, writing just  $\pi_0$  in place of  $\pi_0(\mathcal{G})$ , by base changing to  $\pi_0$ , namely considering the site  $\mathcal{C}/\pi_0$ , every stack  $\mathcal{G}$  is (tautologically) a gerbe over  $\pi_0$  [24].

*Example 2.1.*

<sup>4</sup>"In principle such ideas will lead to a geometric description of all regulator maps, once the categorical aspects have been cleared up. Hopefully this would lead to a better understanding of algebraic K-theory itself."

<sup>5</sup>The automorphisms in  $\text{Aut}(G)$  completely decouple, hence play no role.

- (i) The trivial gerbe with band  $G$  is the stack  $\text{TORS}(G)$  of  $G$ -torsors. Moreover, for any gerbe  $\mathcal{G}$ , the choice of an object  $x$  in  $\mathcal{G}(U)$  determines an equivalence of gerbes  $\mathcal{G}|_U \cong \text{TORS}(G|_U)$ , over  $\mathcal{C}/U$ , where  $G = \text{Aut}_{\mathcal{G}}(x)$ . There is an equivalence  $\text{TORS}(G) \cong \text{B}_G$ , the topos of (left)  $G$ -objects of  $\mathcal{C}^\sim$  ([18]).
- (ii) Any line bundle  $L$  over an algebraic variety  $X$  over  $\mathbb{Q}$  determines a gerbe  $\mathcal{G}_n$  with band  $\mu_n$  (the sheaf of  $n^{\text{th}}$  roots of unity) for any  $n > 1$  as follows: Over any open set  $U$ , consider the category of pairs  $(\mathcal{L}, \alpha)$  where  $\mathcal{L}$  is a line bundle on  $U$  and  $\alpha : \mathcal{L}^{\otimes n} \xrightarrow{\sim} L$  is an isomorphism of line bundles over  $U$ . These assemble to the gerbe  $\mathcal{G}_n$  of  $n^{\text{th}}$  roots of  $L$ . This is an example of a lifting gerbe §2.2.

*Remark.* One also has the following interpretation, which shows that, in a fairly precise sense, a gerbe is the categorical analog of a torsor. Let  $\mathcal{G}$  be a gerbe over  $\mathcal{C}$ , let  $\{U_i\}$  be a cover of  $U \in \text{Ob}(\mathcal{C})$ , and let  $\{x_i\}$  be a collection of objects  $x_i \in \mathcal{G}(U_i)$ . The  $G$ -torsors  $E_{ij} = \text{Hom}(x_j, x_i)$  are part of a “torsor cocycle”  $\gamma_{ijk} : E_{ij} \otimes E_{jk} \rightarrow E_{ik}$ , locally given by  $c_{ijk}$ , above, and subject to the obvious identity. Let  $\text{TORS}(G)$  be the stack of  $G$ -torsors over  $X$ . Since  $G$  is assumed abelian,  $\text{TORS}(G)$  has a group-like composition law given by the standard Baer sum. The fact that  $\mathcal{G}$  itself is locally equivalent to  $\text{TORS}(G)$ , plus the datum of the torsor cocycle  $\{E_{ij}\}$ , show that  $\mathcal{G}$  is equivalent to a  $\text{TORS}(G)$ -torsor.

The primary examples of abelian gerbes occurring in this paper are the gerbe of local lifts associated to a central extension and four-term complexes, described in the next two sections.

**2.2. The gerbe of lifts associated with a central extension.** (See [18, 6, 8].) A central extension

$$(2.2.1) \quad 0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 0$$

of sheaves of groups determines a homotopy-exact sequence

$$\text{TORS}(A) \longrightarrow \text{TORS}(E) \longrightarrow \text{TORS}(G),$$

which is an extension of topoi with characteristic class  $c \in H^2(\text{B}_G, A)$ . (Recall that  $A$  is abelian and that  $\text{TORS}(G)$  is equivalent to  $\text{B}_G$ .) If  $\mathbf{X}$  is any topos over  $\text{TORS}(G) \cong \text{B}_G$ , the gerbe of lifts is the gerbe with band  $A$

$$\mathcal{E} = \text{Hom}_{\text{B}_G}(\mathbf{X}, \text{B}_E),$$

where  $\text{Hom}$  denotes the cartesian morphisms. The class  $c(\mathcal{E}) \in H^2(\mathbf{X}, A)$  is the pullback of  $c$  along the map  $\mathbf{X} \rightarrow \text{B}_G$ . By the universal property of  $\text{B}_G$ , the morphism  $\mathbf{X} \rightarrow \text{B}_G$  corresponds to a  $G$ -torsor  $P$  of  $\mathbf{X}$ , hence the  $A$ -gerbe  $\mathcal{E}$  is the gerbe whose objects are (locally) pairs of the form  $(Q, \lambda)$ , where  $Q$  is an  $E$ -torsor and  $\lambda : Q \rightarrow P$  an equivariant map. It is easy to see that an automorphism of an object  $(Q, \lambda)$  can be identified with an element of  $A$ , so that  $A$  is indeed the band of  $\mathcal{E}$ .

Let us take  $\mathbf{X} = \mathcal{C}^\sim$ , and let  $P$  be a  $G$ -torsor. With the same assumptions as the end of § 2.1, let  $X$  be an object of  $\mathcal{C}$  with a cover  $\{U_i\}$ . In this case, the class of  $\mathcal{E}$  is computed by choosing  $E|_{U_i}$ -torsors  $Q_i$  and equivariant maps  $\lambda_i : Q_i \rightarrow P|_{U_i}$ . Up to refining the cover, let  $\alpha_{ij} : Q_j \rightarrow Q_i$  be an  $E$ -torsor isomorphism such that  $\lambda_i \circ \alpha_{ij} = \lambda_j$ . With these choices the class of  $\mathcal{E}$  is given by the cocycle  $\alpha_{ij} \circ \alpha_{jk} \circ \alpha_{ik}^{-1}$ .

*Remark 2.2.* The above argument gives the well known boundary map [18, Proposition 4.3.4]

$$\partial^1 : H^1(G) \longrightarrow H^2(A)$$

(where we have omitted  $\mathbf{X}$  from the notation). Dropping down one degree we get [ibid., Proposition 3.3.1]

$$\partial^0 : H^0(G) \longrightarrow H^1(A).$$

In fact these are just the boundary maps determined by the above short exact sequence when all objects are abelian. The latter can be specialized even further: if  $g : * \rightarrow G$ , then by pullback the fiber  $E_g$  is an  $A$ -torsor [19].

**2.3. Four-term complexes.** Let  $\mathcal{C}_{\text{ab}}^\sim$  be the category of abelian sheaves over the site  $\mathcal{C}$ . Below we shall be interested in four-term exact sequences of the form:

$$(2.3.1) \quad 0 \longrightarrow A \xrightarrow{i} L_1 \xrightarrow{\partial} L_0 \xrightarrow{p} B \longrightarrow 0.$$

Let  $\text{Ch}_+(\mathcal{C}_{\text{ab}}^\sim)$  be the category of positively graded homological complexes of abelian sheaves. The above sequence can be thought of as a (non-exact) sequence

$$0 \longrightarrow A[1] \longrightarrow [L_1 \longrightarrow L_0] \longrightarrow B \longrightarrow 0$$

of morphisms of  $\text{Ch}_+(\mathcal{C}_{\text{ab}}^\sim)$ . This sequence is short-exact in the sense of Picard categories, namely as a short exact sequence of Picard stacks

$$0 \longrightarrow \text{TORS}(A) \longrightarrow \mathcal{L} \xrightarrow{p} B \longrightarrow 0,$$

where  $\mathcal{L}$  is the strictly commutative Picard stack associated to the complex  $L_1 \rightarrow L_0$  and the abelian object  $B$  is considered as a discrete stack in the obvious way. We have isomorphisms  $A \cong \pi_1(\mathcal{L})$  and  $B \cong \pi_0(\mathcal{L})$ , where the former is the automorphism sheaf of the object  $0 \in \mathcal{L}$  and the latter the sheaf of connected components (see [6, 7, 12]). It is also well known that the projection  $p: \mathcal{L} \rightarrow B$  makes  $\mathcal{L}$  a *gerbe* over  $B$ . In this case the band of  $\mathcal{L}$  over  $B$  is  $A_B$ , thereby determining a class in  $H^2(B, A)$ .<sup>6</sup>

Rather than considering  $\mathcal{L}$  itself as a gerbe over  $B$ , we shall be interested in its fibers above generalized points  $\beta: * \rightarrow B$ . Let us put  $\mathcal{A} = \text{TORS}(A)$ . By a categorification of the arguments in [19], the fiber  $\mathcal{L}_\beta$  above  $\beta$  is an  $\mathcal{A}$ -torsor, hence an  $A$ -gerbe, by the observation at the end of § 2.1 (see also the equivalence described in [5]).  $\mathcal{L}_\beta$  is canonically equivalent to  $\mathcal{A}$  whenever  $\beta = 0$ . Writing

$$\text{Hom}_{\mathcal{C}^\sim}(*, B) \cong \text{Hom}_{\mathcal{C}_{\text{ab}}^\sim}(\mathbb{Z}, B) = H^0(B),$$

we have the homomorphism

$$(2.3.2) \quad \partial^2 : H^0(B) \longrightarrow H^2(A),$$

which sends  $\beta$  to the class of  $\mathcal{L}_\beta$  in  $H^2(A)$ . The sum of  $\beta$  and  $\beta'$  is sent to the Baer sum of  $\mathcal{L}_\beta + \mathcal{L}_{\beta'}$ , and the characteristic class is additive. In the following Lemma we show this map is the same as the one described in [18, Théorème 3.4.2].

**Lemma 2.3.**

- (i) *The map  $\partial^2$  in (2.3.2) is the canonical cohomological map (iterated boundary map) [18, Théorème 3.4.2]*

$$d^2 : H^0(B) \longrightarrow H^1(C) \longrightarrow H^2(A)$$

( $C$  is defined below) arising from the four-term complex (2.3.1).

- (ii) *The image of  $\beta$  under  $d^2$  is the class of the gerbe  $\mathcal{L}_\beta$ .*

*Proof.* We keep the same notation as above. Let us split (2.3.1) as

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & L_1 & \xrightarrow{\pi} & C \longrightarrow 0 \\ & & & & \searrow \partial & \downarrow j & \\ & & & & & L_0 & \\ & & & & & \downarrow p & \\ & & & & & B & \\ & & & & & \downarrow & \\ & & & & & 0 & \end{array}$$

with  $C = \text{Im } \partial$ . By Grothendieck's theory of extensions [19], with  $\beta: * \rightarrow B$ , the fiber  $(L_0)_\beta$  is a  $C$ -torsor (see the end of Remark 2.2). According to section 2.2, we have a morphism  $\text{TORS}(L_1) \rightarrow \text{TORS}(C)$ , and the object  $(L_0)_\beta$  of  $\text{TORS}(C)$  gives rise to the gerbe of lifts  $\mathcal{E}_\beta \equiv \mathcal{E}_{L_0, \beta}$ , which is an  $A$ -gerbe. Now, consider the map assigning to  $\beta \in H^0(B)$  the class of  $\mathcal{E}_\beta \in H^2(A)$ . By construction, this map factors through  $H^1(C)$  by sending  $\beta$  to the class of the torsor  $(L_0)_\beta$ . We then lift that to the class of the gerbe of lifts in  $H^2(A)$ . All stages are compatible with the abelian group structures. This is the homomorphism described in [18, Théorème 3.4.2].

It is straightforward that this is just the classical lift of  $\beta$  through the four-term sequence (2.3.1). Indeed, this is again easily seen in terms of a Čech cover  $\{U_i\}$  of  $*$ . Lifts  $x_i$  of  $\beta|_{U_i}$  are sections of the  $C$ -torsor  $(L_0)_\beta$ , therefore determining a standard  $C$ -valued 1-cocycle  $\{c_{ij}\}$ . From section 2.2 we then obtain an  $A$ -valued 2-cocycle  $\{a_{ijk}\}$  arising from the choice of local  $L_1$ -torsors  $X_i$  such that  $X_i \rightarrow (L_0)_\beta|_{U_i}$  is  $(L_1 \rightarrow C)$ -equivariant.

<sup>6</sup>This is *part* of the invariant classifying the four-term sequence, see the remarks in [7, §6].



Note that in the case at hand,  $\pi: L_1 \rightarrow C$  being an epimorphism, the lifting of the torsor  $(L_0)_\beta$  is done by choosing local trivializations, i.e. the  $x_i$  above, and then choosing  $X_i = L_1|_{U_i}$ .

The same argument shows that the class of  $\mathcal{L}_\beta$ , introduced earlier, is the same as that of  $\mathcal{E}_\beta$ . This follows from the following well known facts: objects of  $\mathcal{L}_\beta$  are locally lifts of  $\beta$  to  $L_0$ ; morphisms between them are given by elements of  $L_1$  acting through  $\partial$ . As a result, automorphisms are sections of  $A$  and clearly the class so obtained coincides with that of  $\mathcal{E}_\beta$ . Therefore  $\mathcal{E}_\beta$  and  $\mathcal{L}_\beta$  are equivalent and the homomorphism of [18, Théorème 3.4.2] is equal to (2.3.2), as required.  $\square$

From the proof of the above lemma, we obtain the following two descriptions of the  $A$ -gerbe  $\mathcal{L}_\beta$ .

**Corollary 2.4.** (i) For any four-term complex (2.3.1) and any generalized point  $\beta$  of  $B$ , the fiber  $\mathcal{L}_\beta$  is a gerbe. Explicitly, it is the stack associated with the prestack which attaches to  $U$  the groupoid  $\mathcal{L}_\beta(U)$  whose objects are elements  $g \in L_0(U)$  with  $p(g) = \beta$  and morphisms between  $g$  and  $g'$  given by elements  $h$  of  $L_1(U)$  satisfying  $\partial(h) = g - g'$ .

(ii) The  $A$ -gerbe  $\mathcal{L}_\beta$  is the lifting gerbe of the  $C$ -torsor  $(L_0)_\beta$  to a  $L_1$ -torsor.  $\square$

We will use both descriptions in §5 especially in the comparison of the Gersten and the Heisenberg gerbe of a codimension two cycle, in the case that it is an intersection of divisors.

A slightly different point of view is the following. Recast the sequence (2.3.1) as a quasi-isomorphism

$$A[2] \xrightarrow{\cong} [L_1 \rightarrow L_0 \rightarrow B]$$

of three-term complexes of  $\text{Ch}_+(\mathcal{C}_{\text{ab}}^\sim)$ , where now  $A$  has been shifted two places to the left. Also, relabel the right hand side as  $L'_2 \rightarrow L'_1 \rightarrow L'_0$  (where again we employ homological degrees) for convenience. By [32], the above morphism of complexes of  $\text{Ch}_+(\mathcal{C}_{\text{ab}}^\sim)$ , placed in degrees  $[-2, 0]$ , gives an equivalence between the corresponding associated strictly commutative Picard 2-stacks

$$\mathfrak{A} \xrightarrow{\cong} \mathfrak{L}$$

over  $\mathcal{C}$ . Here  $\mathfrak{L} = [L'_2 \rightarrow L'_1 \rightarrow L'_0]^\sim$  and  $\mathfrak{A} = [A \rightarrow 0 \rightarrow 0]^\sim \cong \text{Tors}(\mathcal{A}) \cong \text{GERBES}(A)$ . This time we have  $\pi_0(\mathfrak{L}) = \pi_1(\mathfrak{L}) = 0$ , and  $\pi_2(\mathfrak{L}) \cong A$ , as it follows directly from the quasi-isomorphism above. Thus  $\mathfrak{L}$  is 2-connected, namely any two objects are locally (i.e. after base change) connected by an arrow; similarly, any two arrows with the same source and target are—again, locally—connected by a 2-arrow.

Locally, any object of  $\mathfrak{L}$  is a section  $\beta \in B = L'_0$ . By the preceding argument, the Picard stack  $\mathcal{L}_\beta = \text{Aut}_{\mathfrak{L}}(\beta)$  is an  $A$ -gerbe, and the assignment  $\beta \mapsto \mathcal{L}_\beta$  realizes (a quasi-inverse of) the equivalence between  $\mathfrak{A}$  and  $\mathfrak{L}$ . It is easy to see that  $\mathcal{L}_\beta$  is the same as the fiber over  $\beta$  introduced before.

In particular, for the Gersten resolution (5.1.1), (5.1.2), for  $\mathcal{K}_2$ , we get the equivalence of Picard 2-stacks

$$(2.3.3) \quad \text{GERBES}(\mathcal{K}_2) \cong [G_2^X]^\sim.$$

### 3. THE HEISENBERG GROUP

The purpose of this section is to describe a functor  $H: \mathbf{Ab} \times \mathbf{Ab} \rightarrow \mathbf{Grp}$ , where  $\mathbf{Ab}$  is the category of abelian groups and  $\mathbf{Grp}$  that of groups. If  $\mathcal{C}$  is a site, the method immediately generalizes to the categories of abelian groups and of groups in  $\mathcal{C}^\sim$ , the topos of sheaves on  $\mathcal{C}$ . For any pair  $A, B$  of abelian sheaves on  $\mathcal{C}$ , there is a canonical Heisenberg sheaf  $H_{A,B}$  (of non-commutative groups on  $\mathcal{C}$ ), a central extension of  $A \times B$  by  $A \otimes B$ .

The definition of  $H$  is based on a generalization of the Heisenberg group construction due to Brylinski [9, §5]. A pullback along the diagonal map  $A \rightarrow A \otimes A$  gives the extension constructed by Poonen and Rains [28].

**3.1. The Heisenberg group.** Let  $A$  and  $B$  be abelian groups. Consider the (central) extension

$$(3.1.1) \quad 0 \rightarrow A \otimes B \rightarrow H_{A,B} \rightarrow A \times B \rightarrow 0$$

where the group  $H_{A,B}$  is defined by the group law:

$$(3.1.2) \quad (a, b, t)(a', b', t') = (aa', bb', t + t' + a \otimes b').$$

Here  $a, a'$  are elements of  $A$ ,  $b, b'$  of  $B$ , and  $t, t'$  of  $A \otimes B$ . The nonabelian group  $H_{A,B}$  is evidently a functor of the pair  $(A, B)$ , namely a pair of homomorphisms  $(f: A \rightarrow A', g: B \rightarrow B')$  induces a homomorphism  $H_{f,g}: H_{A,B} \rightarrow H_{A',B'}$ . The special case  $A = B = \mu_n$  occurs in Brylinski's treatment of the regulator map to étale cohomology [9].



The map

$$(3.1.3) \quad f: (A \times B) \times (A \times B) \longrightarrow A \otimes B, \quad f(a, b, a', b') = a \otimes b',$$

is a cocycle representing the class of the extension (3.1.1) in  $H^2(A \times B, A \otimes B)$  (group cohomology). Its alternation

$$\varphi_f: \wedge_{\mathbb{Z}}^2(A \times B) \longrightarrow A \otimes B, \quad \varphi_f((a, b), (a', b')) = a \otimes b' - a' \otimes b,$$

coincides with the standard commutator map and represents the value of the projection of the class of  $f$  under the third map in the universal coefficient sequence

$$0 \longrightarrow \text{Ext}^1(A \times B, A \otimes B) \longrightarrow H^2(A \times B, A \otimes B) \longrightarrow \text{Hom}(\wedge_{\mathbb{Z}}^2(A \times B), A \otimes B).$$

As for the commutator map, it is equal to  $[s, s]: \wedge_{\mathbb{Z}}^2(A \times B) \rightarrow A \otimes B$ , where  $s: A \times B \rightarrow H_{A,B}$  is a set-theoretic lift, but the map actually is independent of the choice of  $s$ . (For details see, e.g. the introduction to [7].)

*Remark 3.1.* The properties of the class of the extension  $H_{A,B}$ , in particular that it is a cup-product of the fundamental classes of  $A$  and  $B$ , as we can already evince from (3.1.3), are best expressed in terms of Eilenberg-Mac Lane spaces. We will do this below working in the topos of sheaves over a site.

**3.2. Extension to sheaves.** The construction of the Heisenberg group carries over to the sheaf context. Let  $\mathcal{C}$  be a site, and  $\mathcal{C}^\sim$  the topos of sheaves over  $\mathcal{C}$ . Denote by  $\mathcal{C}_{\text{ab}}^\sim$  the abelian group objects of  $\mathcal{C}^\sim$ , namely the abelian sheaves on  $\mathcal{C}$ , and by  $\mathcal{C}_{\text{grp}}^\sim$  the sheaves of groups on  $\mathcal{C}$ .

For all pairs of objects  $A, B$  of  $\mathcal{C}_{\text{ab}}^\sim$ , it is clear that the above construction of  $H_{A,B}$  carries over to a functor

$$H: \mathcal{C}_{\text{ab}}^\sim \times \mathcal{C}_{\text{ab}}^\sim \longrightarrow \mathcal{C}_{\text{grp}}^\sim.$$

In particular, since  $H_{A,B}$  is already a sheaf of sets (isomorphic to  $A \times B \times (A \otimes B)$ ), the only question is whether the group law varies nicely, but this is clear from its functoriality. Note further that by definition of  $H_{A,B}$  the resulting epimorphism  $H_{A,B} \rightarrow A \times B$  has a global section  $s: A \times B \rightarrow H_{A,B}$  as objects of  $\mathcal{C}^\sim$ , namely  $s = (\text{id}_A, \text{id}_B, 0)$ , which we can use this to repeat the calculations of § 3.1.

In more detail, from § 2.2, the class of the central extension (3.1.1) is to be found in  $H^2(B_{A \times B}, A \otimes B)$  ( $A \otimes B$  is a trivial  $A \times B$ -module). This replaces the group cohomology of § 3.1 with its appropriate topos equivalent. By pulling back to the ambient topos, say  $\mathbf{X} = \mathcal{C}^\sim$ , this is the class of the gerbe of lifts from  $B_{A \times B}$  to  $B_H$ . We are ready to give a proof of Theorem 1.4. This proof is computational.

*Proof of Theorem 1.4.* Let us go back to the cocycle calculations at the end of § 2.2, where  $X$  is an object of  $\mathcal{C}$  equipped with a cover  $\mathcal{U} = \{U_i\}$ . An  $A \times B$ -torsor  $(P, Q)$  over  $X$  would be represented by a Čech cocycle  $(a_{ij}, b_{ij})$  relative to  $\mathcal{U}$ . The cocycle is determined by the choice of isomorphisms  $(P, Q)|_{U_i} \cong (A \times B)|_{U_i}$ . Now, define  $R_i = H_{A,B}|_{U_i}$  with the trivial  $H_{A,B}$ -torsor structure, and let  $\lambda_i: R_i \rightarrow (P, Q)|_{U_i}$  equal the epimorphism in (3.1.1). Carrying out the calculation described at the end of 2.2 with these data gives  $\alpha_{ij} \circ \alpha_{jk} \circ \alpha_{ik}^{-1} = a_{ij} \otimes b_{jk}$ , which is the cup-product in Čech cohomology of the classes corresponding to the  $A$ -torsor  $P$  and the  $B$ -torsor  $Q$ . In other words, the gerbe of lifts corresponding to the central extension determined by the Heisenberg group incarnates the cup product map

$$H^1(X, A) \times H^1(X, B) \xrightarrow{\cup} H^2(X, A \otimes B).$$

For the choice  $\alpha_{ij} = (a_{ij}, b_{ij}, 0)$ , one has the following explicit calculation in the Heisenberg group

$$\begin{aligned} \alpha_{ij} \circ \alpha_{jk} \circ \alpha_{ik}^{-1} &= (a_{ij}, b_{ij}, 0)(a_{jk}, b_{jk}, 0)(a_{ik}, b_{ik}, 0)^{-1} \\ &= (a_{ik}, b_{ik}, a_{ij} \otimes b_{jk})(a_{ik}^{-1}, b_{ik}^{-1}, a_{ik} \otimes b_{ik}) \\ &= (1, 1, a_{ij} \otimes b_{jk} + a_{ik} \otimes b_{ik} - a_{ik} \otimes b_{ik}) \\ &= (1, 1, a_{ij} \otimes b_{jk}); \end{aligned}$$

We used that the inverse of  $(a, b, t)$  in the Heisenberg group is  $(a^{-1}, b^{-1}, -t + a \otimes b)$ :

$$(a, b, t)(a^{-1}, b^{-1}, -t + a \otimes b) = (1, 1, a \otimes b^{-1} + t - t + a \otimes b) = (1, 1, 0).$$

It is well known [8, Chapter 1, §1.3, Equation (1-18), p. 29] that the Čech cup-product of  $a = \{a_{ij}\}$  and  $b = \{b_{ij}\}$  is given by the two-cocycle

$$\{a \cup b\}_{ijk} = \{a_{ij} \otimes b_{jk}\}.$$

This proves the first three points of the statement, whereas the fourth is built-in from the very construction. The fifth follows from the fact that the class of the gerbe of lifts is bilinear: this is evident from the expression computed above.  $\square$

As hinted above, the cup product has a more intrinsic explanation in terms of maps between Eilenberg-Mac Lane objects in the topos. Passing to Eilenberg-Mac Lane objects in particular “explains” why the cup-product realizes the cup-product pairing. First, we state

**Theorem 3.2.** *The class of the extension (3.1.1) in  $\mathcal{C}^\sim$  corresponds to (the homotopy class of) the cup product map*

$$K(A \times B, 1) \cong K(A, 1) \times K(B, 1) \longrightarrow K(A \otimes B, 2)$$

*between the identity maps of  $K(A, 1)$  and  $K(B, 1)$ ; its expression is given by (3.1.3).*

*Proof.* Observe the epimorphism  $H_{A,B} \rightarrow A \times B$  has global set-theoretic sections. The statement follows from Propositions 3.3 and 3.4 below.  $\square$

The two main points, which we now proceed to illustrate, are that Eilenberg-Mac Lane objects represent cohomology (and hypercohomology, once we take into account simplicial objects) in a topos, and that the cohomology of a group object in a topos (such as  $A \times B$  in  $\mathcal{C}^\sim$ ) with trivial coefficients can be traded for the hypercohomology of a simplicial model of it. In this way we calculate the class of the extension as a map, and such map is identified with the cup product. We assemble the necessary results to flesh out the proof of Theorem 3.2 in the next two sections.

**3.3. Simplicial computations.** The class of the central extension (2.2.1) can be computed simplicially. (For the following recollections, see [22, VI.5, VI.6, VI.8] and [4, §2].)

Let  $\mathbb{T}$  be a topos,  $G$  a group-object of  $\mathbb{T}$  (for us it will be  $\mathbb{T} = \mathcal{C}^\sim$ ) and  $BG = K(G, 1)$  the standard classifying simplicial object with  $B_n G = G^n$  [13]. Let  $A$  be a trivial  $G$ -module. We will need the following well known fact.<sup>7</sup>

**Proposition 3.3.**  $H^i(B_G, A) \cong \mathbf{H}^i(BG, A)$ .

*Proof.* The object on the right is the hypercohomology as a simplicial object of  $\mathbb{T}$ . Let  $X$  be a simplicial object in a topos  $\mathbb{T}$ . One defines

$$\mathbf{H}^i(X, A) = \mathbf{Ext}^i(\mathbb{Z}[X]^\sim, A)$$

where  $M^\sim$ , for any simplicial abelian object  $M$  of  $\mathbb{T}$ , denotes the corresponding chain complex defined by  $M_n^\sim = M_n$ , and by taking the alternate sum of the face maps.  $\mathbb{Z}X_n$  denotes the abelian object of  $\mathbb{T}$  generated by  $X_n$ . Of interest to us is the spectral sequence [4, Example (2.10) and below]:

$$E_1^{p,q} = H^q(X_p, A) \implies \mathbf{H}^\bullet(X, A).$$

Let  $X$  be any simplicial object of  $\mathbb{T}$ . The levelwise topoi  $\mathbb{T}/X_n$ ,  $n = 0, 1, \dots$ , form a simplicial topos  $\mathbf{X} = \mathbb{T}/X$  or equivalently a topos fibered over  $\Delta^{\text{op}}$ , where  $\Delta$  is the simplicial category. The topos  $\mathbf{BX}$  of  $\mathbf{X}$ -objects essentially consists of descent-like data, that is, objects  $L$  of  $\mathbf{X}_0$  equipped with an arrow  $a: d_1^* L \rightarrow d_0^* L$  the cocycle condition  $d_0^* a d_2^* a = d_1^* a$  and  $s_0^* a = \text{id}$  (the latter is automatic if  $a$  is an isomorphism). By [22, VI.8.1.3], in the case where  $X = BG$ ,  $\mathbf{BX}$  is nothing but  $\mathbf{B}_G$ , the topos of  $G$ -objects of  $\mathbb{T}$ . One also forms the topos  $\text{Tot}(\mathbf{X})$ , whose objects are collections  $F_n \in \mathbf{X}_n$  such that for each  $\alpha: [m] \rightarrow [n]$  in  $\Delta^{\text{op}}$  there is a morphism  $F_\alpha: \alpha^* F_m \rightarrow F_n$ , where  $\alpha^*$  is the inverse image corresponding to the morphism  $\alpha: \mathbf{X}_n \rightarrow \mathbf{X}_m$ . There is a functor  $\text{ner}: \mathbf{BX} \rightarrow \text{Tot}(\mathbf{X})$  sending  $(L, a)$  to the object of  $\text{Tot}(\mathbf{X})$  which at level  $n$  equals  $(d_0 \cdots d_0)^* L$  ( $a$  enters through the resulting face maps), see *loc. cit.* for the actual expressions. The functor  $\text{ner}$  is the inverse image functor for a morphism  $\text{Tot}(\mathbf{X}) \rightarrow \mathbf{BX}$ , and,  $\mathbf{X}$  satisfying the conditions of being a “good pseudo-category” ([22, VI 8.2]) we have an isomorphism

$$R\Gamma(\mathbf{BX}, L) \xrightarrow{\cong} R\Gamma(\text{Tot}(\mathbf{X}), \text{ner}(L))$$

and, in turn, a spectral sequence

$$E_1^{p,q} = H^q(X_p, \text{ner}_p(L)) \implies H^\bullet(\mathbf{BX}, L),$$

<sup>7</sup>Unfortunately we could not find a specific entry point in the literature to reference, therefore we assemble here the necessary prerequisites. See also [10, §§2,3] for a detailed treatment in the representable case.

[22, VI, Corollaire 8.4.2.2]. On the left hand side we recognize the spectral sequence for the cohomology of a simplicial object in a topos [4, §2.10].

Applying the foregoing to  $X = BG$ , and  $L$  a left  $G$ -object of  $\mathbf{T}$ , we obtain [22, VI.8.4.4.5]

$$E_1^{p,q} = H^q(G^p, L) \implies H^\bullet(B_G, L).$$

(We set  $Y = e$ , the terminal object of  $\mathbf{T}$ , in the formulas from *loc. cit.*)

Thus if  $L = A$ , the trivial  $G$ -module arising from a central extension of  $G$  by  $A$ , by comparing the spectral sequences we can trade  $H^2(B_G, A)$  for the hypercohomology  $\mathbf{H}^2(K(G, 1), A)$ .  $\square$

**3.4. The cup product.** The class of the extension extension (3.1.1) corresponds to the homotopy class of a map  $K(A \times B, 1) \rightarrow K(A \otimes B, 2)$ . We interpret it in terms of cup products of Eilenberg-Mac Lane objects.

Recall that for an object  $M$  of  $\mathbf{C}_{\text{ab}}^\sim$  we have  $K(M, i) = K(M[i])$ , where  $M[i]$  denotes  $M$  placed in homological degree  $i$ , and  $K: \text{Ch}_+(\mathbf{C}_{\text{ab}}^\sim) \rightarrow s\mathbf{C}_{\text{ab}}^\sim$  is the Dold-Kan functor from nonnegative chain complexes of  $\mathbf{C}_{\text{ab}}^\sim$  to simplicial abelian sheaves. Explicitly:

$$K(M, i)_n = \begin{cases} 0 & 0 \leq n < i, \\ \bigoplus_{s: [n] \twoheadrightarrow [i]} M & n \geq i. \end{cases}$$

In particular,  $K(M, i)_i = M$ .  $K$  is a quasi-inverse to the normalized complex functor  $N: s\mathbf{C}_{\text{ab}}^\sim \rightarrow \text{Ch}_+(\mathbf{C}_{\text{ab}}^\sim)$ .

If  $X$  is a simplicial object  $X$  of  $\mathbf{C}^\sim$ , we have

$$(3.4.1) \quad \mathbf{H}^i(X, M) \cong [X, K(M, i)],$$

where the right-hand side denotes the hom-set in the homotopy category [21, 4]. In particular, there is a fundamental class  $\iota_M^n \in \mathbf{H}^n(K(M, n), M)$ , corresponding to the identity map.

Returning to the objects  $A$  and  $B$  of  $\mathbf{C}_{\text{ab}}^\sim$ , also recall the morphism [4, Chapter II, Equation (2.22), p. 64]

$$(3.4.2) \quad \delta_{i,j}: K(A, i) \times K(B, j) \longrightarrow K(A \otimes B, i + j).$$

It is the composition of two maps. The first is:

$$K(A, i) \times K(B, j) \longrightarrow d((K(A, i) \boxtimes K(B, j)) = (K(A, i) \otimes K(B, j))),$$

where  $\boxtimes$  denotes the external tensor product of simplicial objects of  $\mathbf{C}_{\text{ab}}^\sim$  and  $d$  the diagonal; the second is the map in  $s\mathbf{C}_{\text{ab}}^\sim$  corresponding to the Alexander-Whitney map under the Dold-Kan correspondence. We have:

**Proposition 3.4.** *The class of the extension (3.1.1) is equal to  $\iota_A^1 \otimes \iota_B^1 = \delta_{1,1}(\iota_A^1 \times \iota_B^1)$ .*

*Proof.* Observe that any simplicial morphism  $f: X \rightarrow K(M, i)$  is determined by  $f_i$ , the rest, for  $n > i$ , being determined by the simplicial identities. Therefore we need to compute:

$$K(A \times B, 1)_2 \cong K(A, 1)_2 \times K(B, 1)_2 \longrightarrow K(A \otimes B, 2)_2,$$

namely

$$(A \times B) \times (A \times B) \longrightarrow (A \times A) \times (B \times B) \longrightarrow A \otimes B.$$

From the expression of the Alexander-Whitney map, in e.g. [21], the image of the second map in  $\text{Ch}_+(\mathbf{C}_{\text{ab}}^\sim)$  is the sum of  $d_0^v d_0^v$ ,  $d_1^h d_1^h$ , and  $d_2^h d_0^v$ . Only the third one is nonzero, giving  $((a, b), (a', b')) \rightarrow a \otimes b'$ , which equals  $f$  in the construction of the extension (3.1.1). Using (3.4.1) we obtain the conclusion.  $\square$

The morphism (3.4.2) represents the standard cup product in cohomology. By Proposition 3.4, for an object  $X$  of  $s\mathbf{C}^\sim$ , the cup product

$$\mathbf{H}^1(X, A) \times \mathbf{H}^1(X, B) \longrightarrow \mathbf{H}^2(X, A \otimes B)$$

factors through  $X \rightarrow K(A, 1) \times K(B, 1)$  and the extension (3.1.1).

*Remark.* Proposition 3.4 and the above map provide a more conceptual proof of Theorem 1.4.

#### 4. EXAMPLES AND CONNECTIONS TO PRIOR RESULTS

In this section, we collect some examples and briefly indicate the connections with earlier results [2, 9, 27, 28, 30].

**4.1. Self-cup products of Poonen-Rains.** In [28], Poonen and Rains construct, for any abelian group  $A$ , a central extension of the form

$$0 \rightarrow A \otimes A \rightarrow UA \rightarrow A \rightarrow 0,$$

providing a functor  $U: \mathbf{Ab} \rightarrow \mathbf{Grp}$ . The group law in  $UA$  is obtained from (3.1.2) by setting  $a = a'$  and  $b = b'$ . Hence the above extension can be obtained from (3.1.1) by pulling back along the diagonal homomorphism  $\Delta_A: A \rightarrow A \times A$ . Similarly, both the cocycle and its alternation for the extension constructed in *loc. cit.* are obtained from ours by pullback along  $\Delta_A$ , for  $A \in \mathbf{Ab}$ . Similar remarks apply over an abelian sheaf  $A$  on any site  $\mathcal{C}$ . They use  $UA$  to describe the self-cup product  $\alpha \cup \alpha$  of any element  $\alpha \in H^1(A)$ .

**4.2. Brylinski's work on regulators and étale analogues.** In [9], Brylinski has proved Theorem 1.4 in the case  $A = B = \mu_n$ , the étale sheaf  $\mu_n$  of  $n^{\text{th}}$  roots of unity on a scheme  $X$  over  $\text{Spec } \mathbb{Z}[\frac{1}{n}]$  using the Heisenberg group  $H_{\mu_n, \mu_n}$  (in our notation). He used it to provide a geometric interpretation of the regulator map

$$c_{1,2}: H^1(X, \mathcal{K}_2) \longrightarrow H^3(X, \mu_n^{\otimes 2}), \quad (n \text{ odd}),$$

a special case of C. Soulé's regulator. If  $X$  is a smooth projective variety over  $\mathbb{C}$  (viewed as a complex analytic space) and  $f, g$  are invertible functions on  $X$ , P. Deligne (and Bloch) [14] constructed a holomorphic line bundle  $(f, g)$  on  $X$  and Bloch showed that this gives a regulator map from  $K_2(X)$  to the group of isomorphism classes of holomorphic line bundles with connection, later interpreted by D. Ramakrishnan [30] in terms of the three-dimensional Heisenberg group.

Write  $[f]_n, [g]_n \in H^1(X, \mu_n)$  for the images of  $f, g$  under the boundary map  $H^0(X, \mathcal{O}_{X^{an}}) \rightarrow H^1(X, \mu_n)$  of the analytic Kummer sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathcal{O}_{X^{an}}^* \xrightarrow{u \mapsto u^n} \mathcal{O}_{X^{an}}^* \longrightarrow 1.$$

The gerbe  $G_{[f]_n, [g]_n}$  from Theorem 1.4 is compatible with Bloch-Deligne line bundle  $(f, g)$ , in a sense made precise in [9, Proposition 5.1 and after].

**4.3. Finite flat group schemes.** Let  $X$  be any variety over a perfect field  $F$  of characteristic  $p > 0$ . For any commutative finite flat group scheme  $N$  killed by  $p^n$ , consider the cup product pairing

$$H^1(X, N) \times H^1(X, N^D) \rightarrow H^2(X, \mu_{p^n})$$

of flat cohomology groups where  $N^D$  is the Cartier dual of  $N$ . Theorem 1.4 provides a  $\mu_{p^n}$ -gerbe on  $X$  given a  $N$ -torsor and a  $N^D$ -torsor. When  $N$  is the kernel of  $p^n$  on an abelian scheme  $A$  so that  $N^D$  is the kernel of  $p^n$  on the dual abelian scheme  $A^D$  of  $A$ , the cup-product pairing is related to the Néron-Tate pairing [25, p. 19].

**4.4. The gerbe associated with a pair of divisors.** Let  $X$  be a smooth variety over a field  $F$ . Let  $D$  and  $D'$  be divisors on  $X$ . Consider the non-abelian sheaf  $H$  on  $X$  obtained by pushing the Heisenberg group  $H_{\mathcal{K}_1, \mathcal{K}_1}$  along the multiplication map  $m: \mathcal{K}_1 \otimes \mathcal{K}_1 \rightarrow \mathcal{K}_2$ . So  $H$  is a central extension of  $\mathcal{K}_1 \times \mathcal{K}_1$  by  $\mathcal{K}_2$  which we write

$$0 \longrightarrow \mathcal{K}_2 \longrightarrow H \xrightarrow{\pi} \mathcal{K}_1 \times \mathcal{K}_1 \longrightarrow 0.$$

Let  $L = L_{D, D'}$  denote the  $\mathcal{K}_1 \times \mathcal{K}_1$ -torsor defined by the pair  $D, D'$ . Applying Theorem 1.4 gives a  $\mathcal{K}_2$ -gerbe on  $X$  as follows. Since  $H$  is a central extension (so  $\mathcal{K}_1 \times \mathcal{K}_1$  acts trivially on  $\mathcal{K}_2$ ), the category of local liftings of  $L$  to a  $\mathcal{K}_2$ -torsor provide (§2.2, [18, IV, 4.2.2]) a canonical  $\mathcal{K}_2$ -gerbe  $\mathcal{G}_{D, D'}$ .

**Definition 4.1.** *The Heisenberg gerbe  $\mathcal{G}_{D, D'}$  with band  $\mathcal{K}_2$  is the following: For each open set  $U$ , the category  $\mathcal{G}_{D, D'}(U)$  has objects pairs  $(P, \rho)$  where  $P$  is a  $H$ -torsor on  $U$  and*

$$\rho: P \times_{\pi} (\mathcal{K}_1 \times \mathcal{K}_1) \xrightarrow{\sim} L$$

*is an isomorphism of  $\mathcal{K}_1 \times \mathcal{K}_1$ -torsors; a morphism from  $(P, \rho)$  to  $(P', \rho')$  is a map  $f: P \rightarrow P'$  of  $H$ -torsors satisfying  $\rho = \rho' \circ f$ . It is clear that the set of morphisms from  $(P, \rho)$  to  $(P', \rho')$  is a  $\mathcal{K}_2$ -torsor.*

**Example 4.2.** Assume  $X$  is a curve (smooth proper) and put  $Y = X \times X$ .

(i) Assume  $F = \mathbb{F}_q$  is a finite field. Let  $D$  be the graph on  $Y$  of the Frobenius morphism  $\pi: X \rightarrow X$  and  $D'$  be the diagonal, the image of  $X$  under the map  $\Delta: X \rightarrow X \times X$ . Theorem 1.4 attaches a  $\mathcal{K}_2$ -gerbe on  $Y$  to

the zero-cycle  $D.D'$ , the intersection of the divisors  $D$  and  $D'$ . Since the zero cycle  $D.D'$  is the pushforward  $\Delta_*\beta$  of  $\beta = \sum_{x \in X(\mathbb{F}_q)} x$  on  $X$ , we obtain that the set of rational points on  $X$  determines a  $\mathcal{K}_2$ -gerbe on  $X \times X$ .

(ii) Note that the diagonal  $\Delta_Y$  (a codimension-two cycle on  $Y \times Y$ ) can be written as an intersection of divisors  $V$  and  $V'$  on  $Y \times Y = X \times X \times X \times X$  where  $V$  (resp.  $V'$ ) are the set of points of the latter of the form  $\{(a, b, a, c)\}$  (resp.  $\{(a, b, d, b)\}$ ). Theorem 1.4 says that  $\Delta_Y$  determines a  $\mathcal{K}_2$ -gerbe on  $Y \times Y$ .

**4.5. Adjunction formula.** Let  $X$  be a smooth proper variety and  $D$  be a smooth divisor of  $X$ . The classical adjunction formula states:

*The restriction of the line bundle  $L_D^{-1}$  to  $D$  is the conormal bundle  $N_D$  (a line bundle on  $D$ ).*

Given a pair of smooth divisors  $D, D'$  with  $E = D \cap D'$  smooth of pure codimension two, write  $\iota : E \hookrightarrow X$  for the inclusion. There is a map  $\pi : \iota^* \mathcal{K}_2 \rightarrow \mathcal{K}_2^E$ , where  $\mathcal{K}_2^E$  indicates the usual K-theory sheaf  $\mathcal{K}_2$  on  $E$ . An analogue of the adjunction formula for  $E$  would be a description of the  $\mathcal{K}_2^E$ -gerbe  $\pi_* \iota^* \mathcal{G}_{D, D'}$  obtained from the  $\mathcal{K}_2$ -gerbe  $\mathcal{G}_{D, D'}$  on  $X$ .

**Proposition 4.3.** *Let  $D$  and  $D'$  be smooth divisors of  $X$  with  $E = D \cap D'$  smooth of pure codimension two. Consider the line bundles  $V = (N_D)|_E$  and  $V' = (N_{D'})|_E$  on  $E$ . Then,  $\pi_* \iota^* \mathcal{G}_{D, D'}$  is equivalent to the  $\mathcal{K}_2^E$ -gerbe  $\mathcal{G}_{V, V'}$ .*

*Proof.* Since the restriction map  $H^*(X, \mathcal{K}_i) \rightarrow H^*(E, \mathcal{K}_i^E)$  respects cup-product, this follows from the classical adjunction formula for  $D$  and  $D'$ .  $\square$

**4.6. Parshin's adelic groups.** Let  $S$  be a smooth proper surface over a field  $F$ . For any choice of a curve  $C$  in  $S$  and a point  $P$  on  $C$ , Parshin [27, (18)] has introduced a discrete Heisenberg group

$$0 \rightarrow \mathbb{Z} \rightarrow \tilde{\Gamma}_{P, C} \rightarrow \Gamma_{P, C} \rightarrow 0,$$

where  $\Gamma_{P, C}$  is isomorphic (non-canonically) to  $\mathbb{Z} \oplus \mathbb{Z}$ ; he has shown [27, end of §3] how a suitable product of these groups leads to an adelic description of  $CH^2(S)$  and the intersection pairing (1.0.2). His constructions are closely related to an adelic resolution of the sheaf  $H_{\mathcal{K}_1, \mathcal{K}_1}$  on  $S$ .

## 5. ALGEBRAIC CYCLES OF CODIMENSION TWO

Throughout this section,  $X$  is a smooth proper variety over a field  $F$ . Let  $\eta : \text{Spec } F_X \rightarrow X$  be the generic point of  $X$  and write  $K_i^\eta$  for the sheaf  $j_* K_i(F_X)$ .

In this section, we construct the Gersten gerbe  $\mathcal{C}_\alpha$  for any codimension two cycle  $\alpha$  on  $X$ , provide various equivalent descriptions of  $\mathcal{C}_\alpha$  and use them to prove Theorems 5.4, 5.10. As a consequence, we obtain Theorems 1.5 and 1.6 of the introduction.

**5.1. Bloch-Quillen formula.** Recall the (flasque) Gersten resolution<sup>8</sup> [29, §7] [16, p. 276] [17] of the Zariski sheaf  $\mathcal{K}_i$  associated with the presheaf  $U \mapsto K_i(U)$ :

$$(5.1.1) \quad 0 \longrightarrow \mathcal{K}_i \longrightarrow \bigoplus_{x \in X^{(0)}} j_* K_i(x) \longrightarrow \bigoplus_{x \in X^{(1)}} j_* K_{i-1}(x) \longrightarrow \cdots \longrightarrow \bigoplus_{x \in X^{(i-1)}} j_* K_1(x) \xrightarrow{\delta_{i-1}} \bigoplus_{x \in X^{(i)}} j_* K_0(x);$$

here, any point  $x \in X^{(m)}$  corresponds to a subvariety of codimension  $m$  and the map  $j$  is the canonical inclusion  $x \hookrightarrow X$ . So  $\mathcal{K}_i$  is quasi-isomorphic to the complex

$$(5.1.2) \quad G_i^X = [K_i^\eta \longrightarrow \bigoplus_{x \in X^{(1)}} j_* K_{i-1}(x) \longrightarrow \cdots \longrightarrow \bigoplus_{x \in X^{(i-1)}} j_* K_1(x) \xrightarrow{\delta_{i-1}} \bigoplus_{x \in X^{(i)}} j_* K_0(x)].$$

By (5.1.1), there is a functorial isomorphism [29, §7, Theorem 5.19] [16, Corollary 72, p. 276]

$$(5.1.3) \quad \bigoplus_i CH^i(X) \xrightarrow{\sim} \bigoplus_i H^i(X, \mathcal{K}_i); \quad (\text{Bloch-Quillen formula})$$

this is an isomorphism of graded rings: D. Grayson has proved that the intersection product on  $CH(X) = \bigoplus_i CH^i(X)$  corresponds to the cup-product in cohomology [16, Theorem 77, p.278]. Thus, algebraic cycles of codimension  $n$  give  $n$ -cocycles of the sheaf  $\mathcal{K}_n$  on  $X$  and that two such cocycles are cohomologous exactly when the algebraic cycles are rationally equivalent.

<sup>8</sup>This resolution exists for any smooth variety over  $F$ .

The final two maps in (5.1.1) arise essentially from the valuation and the tame symbol map [2, pp.351-2]. Let  $R$  be a discrete valuation ring, with fraction field  $L$ ; let  $\text{ord} : L^\times \rightarrow \mathbb{Z}$  be the valuation and let  $l$  be the residue field. The boundary maps from the localization sequence for  $\text{Spec } R$  are known explicitly: the map  $L^\times = K_1(L) \rightarrow K_0(l) = \mathbb{Z}$  is the map  $\text{ord}$  and the map  $K_2(L) \rightarrow K_1(l) = l^\times$  is the tame symbol. This applies for any normal subvariety  $V$  (corresponds to a  $y \in X^{(i)}$ ) and a divisor  $x$  of  $V$  (corresponding to a  $x \in X^{(i+1)}$ ).

**5.2. Divisors.** We recall certain well known results about divisors and line bundles for comparison with the results below for the  $\mathcal{K}_2$ -gerbes attached to codimension two cycles.

If  $A$  is a sheaf of abelian groups on  $X$ , then  $\text{Ext}_X^1(\mathbb{Z}, A) = H^1(X, A)$  classifies  $A$ -torsors on  $X$ . Given an extension  $E$

$$0 \longrightarrow A \longrightarrow E \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0$$

of abelian sheaves on  $X$ , the corresponding  $A$ -torsor is simply  $\pi^{-1}(1)$  (a sheaf of sets). When  $X$  is a point, then  $\pi^{-1}(1)$  is a coset of  $\pi^{-1}(0) = A$ , i.e., a  $A$ -torsor. The classical correspondence [20] between Weil divisors (codimension-one algebraic cycles)  $D$  on  $X$ , Cartier divisors, line bundles  $\mathcal{L}_D$ , and torsors  $\mathcal{O}_D$  over  $\mathcal{O}_X^* = \mathbb{G}_m = \mathcal{K}_1$  comes from the Gersten sequence (5.1.1) for  $\mathcal{K}_1$  (see also [17, 2.2]):

$$(5.2.1) \quad 0 \longrightarrow \mathcal{O}_X^* \longrightarrow F_X^\times \xrightarrow{d} \bigoplus_{x \in X^{(1)}} j_* \mathbb{Z} \longrightarrow 0,$$

where  $F_X$  is the constant sheaf of rational functions on  $X$  and the sum is over all irreducible effective divisors on  $X$ , using that  $K_0(L) \cong \mathbb{Z}$  and  $K_1(L) = L^\times$  for any field  $L$ . As a Weil divisor  $D = \sum_{x \in X^{(1)}} n_x x$  is a formal combination with integer coefficients of subvarieties of codimension one of  $X$ , it determines a map of sheaves

$$\psi : \mathbb{Z} \longrightarrow \bigoplus_{x \in X^{(1)}} j_* \mathbb{Z};$$

$\psi(1)$  is the section with components  $n_x$ . The  $\mathcal{O}_X^*$ -torsor  $\mathcal{O}_D$  attached to  $D$  is given as the subset

$$(5.2.2) \quad d^{-1}(\psi(1)) \subset F_X^\times.$$

A Čech description of  $\mathcal{O}_D$  relative to an Zariski open cover  $\{U_i\}$  of  $X$  is as follows. Pick a rational function  $f_i$  on  $U_i$  with pole of order  $n_x$  along  $x$  for all  $x \in U_i^{(1)}$  (so  $x$  is a irreducible subvariety of codimension one of  $U_i$ ); we view  $f_i \in F_X^\times$ . On  $U_i \cap U_j$ , one has  $f_i = g_{ij} f_j$  for unique  $g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$ ; the collection  $\{g_{ij}\}$  is a Čech one-cocycle with values in  $\mathcal{O}_X^*$  representing  $\mathcal{O}_D$ .

For any  $D$ ,  $\mathcal{L}_D$  is trivial on the complement of the support of  $D$ .

*Remark 5.1.* For each open  $U$  of  $X$ , one has the Picard category  $\text{TORS}_U(\mathcal{O}^*)$  of  $\mathcal{O}^*$ -torsors on  $U$ . These combine to the Picard stack  $\text{TORS}(\mathcal{O}^*)$  of  $\mathcal{O}^*$ -torsors on  $X$ . The Gersten sequence incarnates this Picard stack [11, 1.10].  $\square$

**5.3. The Gersten gerbe of a codimension two cycle.** We next show that every cycle  $\alpha$  of codimension two on  $X$  determines a gerbe  $\mathcal{C}_\alpha$  with band  $\mathcal{K}_2$ . The Gersten complex (5.1.1) enables us to give a geometric description of  $\mathcal{C}_\alpha$ ; see Remark 5.5 below.

The cycle  $\alpha$  provides a natural map

$$(5.3.1) \quad 0 \longrightarrow \mathcal{K}_2 \xrightarrow{\mu} K_2^\eta \xrightarrow{\nu} \bigoplus_{x \in X^{(1)}} j_* K_1(x) \xrightarrow{\delta} \bigoplus_{x \in X^{(2)}} j_* K_0(x) \longrightarrow 0$$

$\begin{array}{c} \mathbb{Z} \\ \downarrow \phi \end{array}$

and an exact sequence (by pullback)

$$(5.3.2) \quad 0 \longrightarrow \mathcal{K}_2 \longrightarrow K_2^\eta \xrightarrow{\nu} T \xrightarrow{\delta} \mathbb{Z} \longrightarrow 0.$$

This two-extension of  $\mathbb{Z}$  by  $\mathcal{K}_2$  gives a class in  $\text{Ext}^2(\mathbb{Z}, \mathcal{K}_2) = H^2(X, \mathcal{K}_2)$ . Writing  $\alpha = \sum_x n_x [x]$  as a sum over  $x \in X^{(2)}$  (irreducible codimension two subvarieties), then the  $x$ -component of  $\phi(1)$  corresponds to  $n_x$  under the canonical isomorphism  $K_0(x) \cong \mathbb{Z}$ . The maps  $\delta$  and  $\nu$  are essentially given by the valuation (or ord) and tame symbol maps; see §5.1.



**Definition 5.2.** The gerbe  $\mathcal{C}_\alpha$  (associated with the cycle  $\alpha$ ) is obtained by applying the results of §2.3 to (5.3.1), (5.3.2); thus it is an example of the gerbe  $\mathcal{L}_\beta$  of §2.3, where  $\beta = \phi$  and  $\mathcal{L}$  is the Picard stack associated to the complex  $[K_2^\eta \rightarrow \bigoplus_{x \in X^{(1)}} j_* K_1(x)]$ .

*Remark 5.3.* Corollary 2.4 provides two descriptions of  $\mathcal{L}_\beta$ . It should be emphasized that both descriptions are useful. One of them, which we make explicit below, is crucial for the comparison with the Heisenberg gerbe (Theorem 5.10); the other succinct description is given in Remark 5.5.

- (i) For any open set  $U$  of  $X$ , the category  $\mathcal{C}_\alpha(U)$  has objects  $u \in \bigoplus_{x \in X^{(1)}} j_* K_1(x)$  with  $\delta u = \phi(1)$  and morphisms from  $u$  to  $u'$  are elements  $a \in K_2^\eta$  with  $\nu(a) = u' - u$ .
- (ii) Any Hom-set  $\text{Hom}_{\mathcal{C}_\alpha}(u, u')$  is a  $K_2(U)$ -torsor.
- (iii) The category  $\mathcal{C}_\alpha(U)$  can be described geometrically in terms of the ord and tame maps. For instance, let  $X$  be a surface. Write the zero-cycle  $\alpha$  as a finite sum  $\sum_{i \in I} n_i x_i$  of points  $x_i$  of  $X$ . We assume  $n_i \neq 0$  and write  $V$  for the complement of the support of  $\alpha$ . Any non-zero rational function  $f$  on a curve  $C$  defines an object of  $\mathcal{C}_\alpha(U)$  if  $f$  is invertible on  $C \cap U \cap V$  and satisfies  $\text{ord}_{x_i} f = n_i$  for each  $x_i \in U$  (assuming, for simplicity, that  $x_i$  is a smooth point of  $C$ ). A general object of  $\mathcal{C}_\alpha(U)$  is a finite collection  $u = \{C_j, f_j\}$  of curves  $C_j$  and non-zero rational functions  $f_j$  on  $C_j$  such that  $f_j$  is invertible on  $C_j \cap U \cap V$  and  $\sum \text{ord}_{x_i} f_j = n_i$  (an index  $j$  occurs in the sum if  $x_i \in C_j$ ) for each  $x_i \in U$ . A morphism from  $u$  to  $u'$  is an element  $a \in K_2^\eta$  whose tame symbol is  $u' - u$ .  $\square$

**Theorem 5.4.** (i)  $\mathcal{C}_\alpha$  is a gerbe on  $X$  with band  $\mathcal{K}_2$ .

(ii) Under (5.1.3), the class of  $\mathcal{C}_\alpha \in H^2(X, \mathcal{K}_2)$  corresponds to  $\alpha \in CH^2(X)$ .

(iii)  $\mathcal{C}_\alpha$  is equivalent to  $\mathcal{C}_{\alpha'}$  (as gerbes) if and only if the cycles  $\alpha$  and  $\alpha'$  are rationally equivalent.

*Proof.* (i) The Gersten sequence (5.3.1) is an example of a four-term complex, discussed in §2.3. As the stack  $\mathcal{C}_\alpha$  is a special case of the gerbe  $\mathcal{L}_\beta$  constructed in §2.3, (i) is obvious.

In more detail: We first observe that (5.3.2) provides a quasi-isomorphism between  $\mathcal{K}_2$  (sheaf) and the complex (concentrated in degree zero and one)

$$(5.3.3) \quad \eta : \mathcal{K}_2 \rightarrow [K_2^\eta \xrightarrow{\nu} \text{Ker}(\delta)].$$

Now, suppose  $U$  is disjoint from the support of  $\alpha$ . On such an open set  $U$ , the map  $\phi$  is zero. This means that the objects  $u$  of the category  $\mathcal{C}_\alpha(U)$  are elements of  $\text{Ker}(\delta)$ . The gerbe  $\mathcal{C}_\alpha$ , when restricted to  $U$ , is equivalent to the Picard stack of  $\mathcal{K}_2$ -torsors [1, Expose XVIII, 1.4.15]: in the complex  $[K_2^\eta \xrightarrow{\nu} \text{Ker}(\delta)]$ , one has  $\text{Coker}(\nu) = 0$  and  $\text{Ker}(\nu) = \mathcal{K}_2|_U$ . Since for any abelian sheaf  $G$ , the category  $\text{TORS}(G)$  is the trivial  $G$ -gerbe,  $\mathcal{C}_\alpha$  is the trivial gerbe with band  $\mathcal{K}_2$  on the complement of the support of  $\alpha$ .

Now, consider an arbitrary open set  $V$  of  $X$ . By the exactness of (5.3.2), there is an open covering  $\{U_i\}$  of  $V$  and sections  $u_i \in T(U_i)$  with  $t_i = \phi(1)$ . Fix  $i$  and let  $U$  be an open set contained in  $U_i$ . Then the category  $\mathcal{C}_\alpha(U)$  is non-empty. The category  $D$  with objects  $d \in \text{Ker}(\delta) \subset T(U)$  and morphisms  $\text{Hom}_D(d, d') = \{a \in K_2^\eta \mid \nu(a) = d' - d\}$  is clearly equivalent to the category of  $K_2(U)$ -torsors. The functor which sends  $d$  to  $d + u_i$  is easily seen to be an equivalence of categories between  $D$  and  $\mathcal{C}_\alpha(U)$ . Thus  $\mathcal{C}_\alpha$  is a gerbe with band  $\mathcal{K}_2$ .

(ii) The Bloch-Quillen formula (5.1.3) arises from the canonical map

$$d^2 : Z^2(X) \rightarrow H^2(X, \mathcal{K}_2)$$

of Lemma 2.3 attached to the four-term complex (5.3.1). As  $\mathcal{C}_\alpha$  is a gerbe of the form  $\mathcal{L}_\beta$ , (ii) follows from Lemma 2.3.

(iii) This is a simple consequence of the Bloch-Quillen formula (5.1.3).  $\square$

*Remark 5.5.* (i) Split the sequence (5.3.1) into

$$0 \longrightarrow \mathcal{K}_2 \longrightarrow K_2^\eta \longrightarrow K_2^\eta / \mathcal{K}_2 \longrightarrow 0$$

and

$$0 \longrightarrow K_2^\eta / \mathcal{K}_2 \longrightarrow \bigoplus_{x \in X^{(1)}} j_* K_1(x) \longrightarrow \bigoplus_{x \in X^{(2)}} j_* K_0(x) \longrightarrow 0.$$

Since the Gersten resolution is by flasque sheaves, one has  $H^1(X, K_2^\eta / \mathcal{K}_2) \xrightarrow{\sim} H^2(X, \mathcal{K}_2)$ . As Cartier divisors are elements of  $H^0(X, K_1^\eta / \mathcal{K}_1)$ , we view elements of  $H^1(X, K_2^\eta / \mathcal{K}_2)$  as *Cartier cycles of codimension*

two. The map  $Z^1(X) \rightarrow H^1(X, K_2^\eta/\mathcal{K}_2)$  attaches to any cycle its Cartier cycle. Lemma 2.4 provides the following succinct description of  $\mathcal{C}_\alpha$ :

*it is the gerbe of liftings (to a  $K_2^\eta$ -torsor) of the  $(K_2^\eta/\mathcal{K}_2)$ -torsor determined by  $\alpha$ .*

(iii) The proof of Theorem 5.4 provides a canonical trivialization<sup>9</sup>  $\eta_\alpha$  of the gerbe  $\mathcal{C}_\alpha$  on the complement of the support of  $\alpha$ .

(iv) The pushforward of  $\mathcal{C}_\alpha$  along  $\mathcal{K}_2 \rightarrow \Omega^2$  produces a  $\Omega^2$ -gerbe which manifests the cycle class of  $\alpha$  in de Rham cohomology  $H^2(X, \Omega^2)$ . If  $\alpha$  is homologically equivalent to zero, then this latter gerbe is trivial, i.e., it is the Picard stack of  $\Omega^2$ -torsors.  $\square$

*Remark 5.6.* It may be instructive to compare the  $\mathbb{G}_m$ -torsor  $\mathcal{O}_D$  attached to a divisor  $D$  of  $X$  and the  $\mathcal{K}_2$ -gerbe  $\mathcal{C}_\alpha$  attached to a codimension-two cycle. Let  $U$  be any open set of  $X$ . This goes, roughly speaking, as follows.

- $\mathcal{O}_D$ : The set of divisors on  $U$  rationally equivalent to zero is exactly the image of  $d$  over  $U$  in (5.2.1). So, the set  $\mathcal{O}_D(U)$  is non-empty if  $D = 0$  in  $CH^1(U)$ . The sections of  $\mathcal{O}_D$  over  $U$  are given by rational functions  $f$  on  $U$  whose divisor is  $D|_U$ . In other words, the sections are rational equivalences between the divisor  $D$  and the empty divisor. The set  $\mathcal{O}_D(U)$  is a torsor over  $\mathbb{G}_m(U)$ .
- $\mathcal{C}_\alpha$ : We observe that the image of  $\delta$  in (5.3.1) consists of codimension-two cycles rationally equivalent to zero. So  $\mathcal{C}_\alpha$  is non-empty if  $\alpha = 0$  in  $CH^2(U)$ . Each rational equivalence between  $\alpha$  and the empty codimension-two cycle gives an object of  $\mathcal{C}_\alpha(U)$ . The sheaf of morphisms between two objects is a  $\mathcal{K}_2$ -torsor.  $\square$

The Bloch-Quillen formula (5.1.3) states that equivalence classes of  $\mathcal{K}_2$ -gerbes are in bijection with codimension-two cycles (modulo rational equivalence) on  $X$ . We have seen that a codimension-two cycle determines a  $\mathcal{K}_2$ -gerbe (an actual gerbe, not just one up to equivalence). It is natural to ask whether the converse holds: (see Proposition 5.8 in this regard)

**Question 5.7.** *Does a  $\mathcal{K}_2$ -gerbe on  $X$  determine an actual codimension-two cycle?*

Consider the  $\mathcal{K}_2$ -gerbe  $\mathcal{G}_{D,D'}$  attached to a pair of divisors  $D, D'$  on  $X$ . If  $\mathcal{G}_{D,D'}$  determines an actual codimension-two cycle, then any pair  $D, D'$  of divisors determines a canonical codimension-two cycle on  $X$ . This implies that there is a canonical intersection of Weil divisors and this last statement is known to be false. So the answer to Question 5.7 is negative in general.

**5.4. Gerbes and cohomology with support.** Let  $F$  be an abelian sheaf on a site  $\mathcal{C}$ . Recall that (see e.g. [26, §5.1])  $H^1(F)$  is the set of isomorphism classes of auto-equivalences of the trivial gerbe  $\text{TORS}(F)$  with band  $F$ ; more generally, given gerbes  $\mathcal{G}$  and  $\mathcal{G}'$  with band  $F$ , then the set  $\text{Hom}_{\mathcal{C}}(\mathcal{G}, \mathcal{G}')$  (assumed non-empty) of maps of gerbes is a torsor for  $H^1(F)$ .

Recall also that, for any sheaf  $F$  on a scheme  $V$ , the cohomology  $H_Z^*(V, F)$  with support in a closed subscheme  $Z$  of  $V$  fits into an exact sequence [2, §5]

$$(5.4.1) \quad \cdots \longrightarrow H_Z^i(V, F) \longrightarrow H^i(V, F) \longrightarrow H^i(V - Z, F) \longrightarrow H_Z^{i+1}(V, F) \longrightarrow \cdots;$$

the exactness of

$$H^1(V, F) \longrightarrow H^1(V - Z, F) \longrightarrow H_Z^2(V, F) \longrightarrow H^2(V, F) \longrightarrow H^2(V - Z, F)$$

leads to an interpretation of the group  $H_Z^2(V, F)$ : it classifies isomorphism classes of pairs  $(\mathcal{G}, \phi)$  consisting of a gerbe  $\mathcal{G}$  with band  $F$  on  $V$  and a trivialization  $\phi$  of  $\mathcal{G}$  on  $V - Z$ , i.e.,  $\phi$  is an equivalence of  $\mathcal{G}|_{V-Z}$  with  $\text{TORS}(F|_{V-Z})$ .

**5.5. Geometric interpretation of some results of Bloch.** Bloch [2] has proved that:

- [2, Proposition 5.3] Any codimension-two cycle  $\alpha$  on  $X$  has a canonical cycle class  $[\alpha] \in H_Z^2(X, \mathcal{K}_2)$ ; here  $Z$  is the support of  $\alpha$ .
- [2, Theorem 5.11] If  $D$  is a smooth divisor of  $X$ , then  $\text{Pic}(D) = H^1(D, \mathcal{K}_1)$  is a direct summand of  $H_D^2(X, \mathcal{K}_2)$ .

<sup>9</sup>This uses (5.3.3).

For (1), we note that, by Remark 5.5, the gerbe  $\mathcal{C}_\alpha$  has a trivialization  $\eta_\alpha$  on  $X - Z$ . By the above interpretation of  $H^2$  with support, the pair  $(\mathcal{C}_\alpha, \eta_\alpha)$  defines an element of  $H_Z^2(X, \mathcal{K}_2)$ ; this is the canonical class  $[\alpha]$ .

For (2), recall that Bloch constructed maps  $a : \text{Pic}(D) \rightarrow H_D^2(X, \mathcal{K}_2)$  and  $b : H_D^2(X, \mathcal{K}_2) \rightarrow \text{Pic}(D)$  with  $b \circ a$  the identity on  $\text{Pic}(D)$ . We can interpret the map  $a$  as follows. Note that any divisor  $E$  of  $D$  is a codimension-two cycle  $\alpha$  on  $X$ . The  $\mathcal{K}_2$ -gerbe  $\mathcal{C}_\alpha$  on  $X$  has a canonical trivialization  $\eta_\alpha$  on  $X - E$  (and so also on the smaller  $X - D$ ). The association  $E \mapsto (\mathcal{C}_\alpha, \eta_\alpha)$  gives the homomorphism  $a : \text{Pic}(D) \rightarrow H_D^2(X, \mathcal{K}_2)$ .

These results of Bloch provide a partial answer to Question 5.7 summarized in the following

**Proposition 5.8.** *Let  $\mathcal{G}$  be a  $\mathcal{K}_2$ -gerbe on  $X$  and let  $\beta \in CH^2(X)$  correspond to  $\mathcal{G}$  in the Bloch-Quillen formula (5.1.3). Let  $\phi$  be a trivialization of  $\mathcal{G}$  on the complement  $X - D$  of a smooth divisor  $D$  of  $X$ . Then,  $\beta$  can be represented by a divisor of  $D$  (unique up to rational equivalence on  $D$ ).*

Note that the data of  $\phi$  is crucial: the map  $\text{Pic}(D) \rightarrow CH^2(X)$  is not injective in general [3, (iii), p. 269].

**Proposition 5.9.** *Let  $i : D \rightarrow X$  and  $j : U = X - D \rightarrow X$  be the inclusion maps. We have the following short exact sequence of Picard 2-stacks*

$$\text{TORS}(i_*\mathcal{K}_1^D) \longrightarrow \text{GERBES}(\mathcal{K}_2^X) \longrightarrow \text{GERBES}(j_*\mathcal{K}_2^U).$$

*Proof.* Analyzing the Gersten sequence (5.1.1), (5.1.2) for  $\mathcal{K}_2$  on  $X$  and  $U$ , we get the short exact sequence:

$$0 \longrightarrow i_*G_1^D \longrightarrow G_2^X \longrightarrow j_*G_2^U \longrightarrow 0.$$

This gives a short exact sequence of Picard 2-stacks, then use (2.3.3). Note that  $\text{TORS}(i_*\mathcal{K}_1^D)$  is considered as a Picard 2-stack with no nontrivial 2-morphisms.  $\square$

The global long exact cohomology sequence arising from the exact sequence in the proposition gives part of the localization sequence for higher Chow groups

$$\cdots \longrightarrow CH^1(D, 1) \longrightarrow CH^2(X, 1) \longrightarrow CH^2(X - D, 1) \longrightarrow \text{Pic}(D) \longrightarrow CH^2(X) \longrightarrow CH^2(X - D) \longrightarrow 0.$$

This uses the fact that  $CH^1(D, 0) = \text{Pic}(D)$ , that  $CH^1(D, 1) = H^0(D, \mathcal{O}^*)$  and  $CH^1(D, j)$  is zero for  $j > 1$  [3, (viii), p. 269].

**5.6. The two gerbes associated with an intersection of divisors.** For a codimension-two cycle of  $X$  presented as the intersection of divisors, we know that the  $\mathcal{K}_2$ -gerbes in Theorem 5.4 (Gersten gerbe) and in §4.4 (using Theorem 1.4) (Heisenberg gerbe) are equivalent (as their class in  $H^2(X, \mathcal{K}_2)$  corresponds to the class of the codimension-two cycle in  $CH^2(X)$  via (5.1.3)). We now construct an actual equivalence between them.

**Theorem 5.10.** *Suppose that the codimension-two cycle  $\alpha$  is the intersection  $D \cdot D'$  of divisors  $D$  and  $D'$  on  $X$ . There is a natural equivalence<sup>10</sup>*

$$\Theta : \mathcal{C}_\alpha \rightarrow \mathcal{G}_{D, D'}$$

*of  $\mathcal{K}_2$ -gerbes on  $X$ .*

*Proof.* By Theorem 1.4 and Theorem 5.4, the classes of the gerbes  $\mathcal{G}_{D, D'}$  and  $\mathcal{C}_\alpha$  in  $H^2(X, \mathcal{K}_2)$  both correspond to the class of  $\alpha$  in  $CH^2(X)$ . This shows that they are equivalent.

Let us exhibit an actual equivalence. We will construct a functor  $\Theta_U : \mathcal{C}_\alpha(U) \rightarrow \mathcal{G}_{D, D'}(U)$ , compatible with restriction maps  $V \subset U \subset X$ .

Consider an object  $r \in \mathcal{C}_\alpha(U)$ . We want to attach to  $r$  a  $H$ -torsor  $\Theta_U(r)$  on  $U$  in a functorial manner. Each  $\Theta_U(r)$  is a  $H$ -torsor which lifts the  $\mathcal{K}_1 \times \mathcal{K}_1$ -torsor  $\mathcal{O}_D \times \mathcal{O}_{D'}$  on  $U$ . We will describe  $\Theta_U(r)$  by means of Čech cocycles. Fix an open covering  $\{U_i\}$  of  $U$  and write  $\mathcal{C}^n(A)$  for Čech  $n$ -cochains with values in the sheaf  $A$  with respect to this covering.

<sup>10</sup>By §5.4, the set of such equivalences is a torsor over  $H^1(X, \mathcal{K}_2) = CH^2(X, 1)$  [31, §2.1].

*Step 1.* Let  $a = \{a_{i,j}\}$  and  $b = \{b_{i,j}\}$  with  $a, b \in \mathcal{C}^1(O^*)$  be Čech 1-cocycles for  $\mathcal{O}_D$  and  $\mathcal{O}_{D'}$ . Pick  $h = \{h_{i,j}\} \in \mathcal{C}^1(H)$  of the form

$$h_{i,j} = (a_{i,j}, b_{i,j}, c_{i,j}) \in H(U_i \cap U_j).$$

We need  $c_{i,j} \in K_2(U_i \cap U_j)$  such that  $h$  is a Čech 1-cocycle (for  $\Theta_U(r)$ , the putative  $H$ -torsor). Since  $a, b$  are Čech cocycles, the Čech boundary  $\partial h$  is of the form

$$\partial h = \{(1, 1, y_{i,j,k})\}$$

with  $y = \{y_{i,j,k}\} \in \mathcal{C}^2(K_2)$  a Čech 2-cocycle. This cocycle  $y$  represents the gerbe  $\mathcal{G}_{D,D'}$  on  $U$ .

*Step 2.* Recall that  $\mathcal{C}_\alpha$  is the associated stack of the prestack  $U \mapsto \mathcal{C}_\alpha(U)$  where the category  $\mathcal{C}_\alpha(U)$  has objects  $u \in \oplus_{x \in X^1} j_* K_1(x)$  with  $\delta u = \phi(1)$  and morphisms from  $u$  to  $v$  are elements  $a \in K_2^\eta$  with  $\nu(a) = v - u$ . Since the category  $\mathcal{C}_\alpha(U)$  is non-empty, the class of the gerbe  $\mathcal{C}_\alpha$  (restricted to  $U$ ) in  $H^2(U, K_2)$  is zero. Since  $\mathcal{C}_\alpha$  and  $\mathcal{G}_{D,D'}$  are equivalent, so the class of  $\mathcal{G}_{D,D'}$  in  $H^2(U, K_2)$  is also zero.

*Step 3.* Consider the case  $r$  is given by a pair  $(C, g)$  where  $C$  is a divisor on  $X$  and  $g$  is a rational function on  $C$ . The condition  $\delta(r) = \phi(1)$  says  $\alpha \cap U$  is the intersection of  $U$  with the zero locus of  $g$ . Assume  $g \in \mathcal{O}_C(C \cap U)$ . Given any lifting  $\tilde{g} \in \mathcal{O}_X(U)$  with divisor  $C'$  on  $U$ , we can write  $\alpha \cap U$  as the intersection of the divisors  $C \cap U$  and the (principal) divisor  $C'$  in  $U$ . By the results in §4.4, there is a  $K_2$ -gerbe  $\mathcal{G}_{C \cap U, C'}$  on  $U$ . As  $C'$  is principal, its class in  $H^1(U, K_1)$  is zero; so the class of  $\mathcal{G}_{C \cap U, C'}$  in  $H^2(U, K_2)$  is zero.

*Step 4.* Let  $z = \{z_{i,j,k}\} \in \mathcal{C}^2(K_2)$  be a Čech 2-cocycle for  $\mathcal{G}_{C \cap U, C'}$ ;

So  $z = \partial w$  is the boundary of a Čech cochain  $w = \{w_{i,j}\} \in \mathcal{C}^1(K_2)$ . Note that  $y - z = \partial v$  for a 1-cochain  $v$  since  $\mathcal{G}_{C \cap U, C'}$  and  $\mathcal{G}_{D,D'}$  are equivalent as gerbes on  $U$ : both are trivial on  $U$ !

The Čech cochain  $h' = \{h'_{i,j}\} \in \mathcal{C}^1(H)$  with

$$h'_{i,j} = (a_{i,j}, b_{i,j}, c_{i,j})(1, 1, -w_{i,j})(1, 1, -v_{i,j})$$

is a Čech cocycle and represents the required  $H$ -torsor  $\Theta_U(r)$  on  $U$ .

*Step 5.* The same argument with simple modifications works for a general object of  $\mathcal{C}_\alpha$ . It is easy to check that  $\Theta_U$  is a functor, compatible with restriction maps  $V \subset U \subset X$ , and that the induced morphism of gerbes is an equivalence.  $\square$

**5.7. Higher gerbes attached to smooth Parshin chains.** By Gersten's conjecture, the localization sequence [29, §7 Proposition 3.2] breaks up into short exact sequences

$$0 \longrightarrow K_i(V) \longrightarrow K_i(V - Y) \longrightarrow K_{i-1}(Y) \longrightarrow 0, \quad (i > 0)$$

for any smooth variety  $V$  over  $F$  and a closed smooth subvariety  $Y$  of  $V$ . Let  $j : D \rightarrow X$  be a smooth closed subvariety of codimension one of  $X$ ; write  $\iota : X - D \rightarrow X$  for the open complement of  $D$ . Any divisor  $\alpha$  of  $D$  is a codimension-two cycle on  $X$ ; one has a map  $\text{Pic}(D) \rightarrow CH^2(X)$  [3, (iii), p. 269]. This gives the exact sequence (for  $i > 0$ )

$$0 \longrightarrow \mathcal{K}_i \longrightarrow \mathcal{F}_i \longrightarrow j_* \mathcal{K}_{i-1}^D \longrightarrow 0$$

of sheaves on  $X$  where  $\mathcal{F}_i = \iota_* \mathcal{K}_i^U$  is the sheaf associated with the presheaf  $U \mapsto K_i(U - D)$ . We write  $\mathcal{K}_i^D$  and  $\mathcal{K}_i^U$  for the usual K-theory sheaves on  $D$  and  $U$  since the notation  $\mathcal{K}_i$  is already reserved for the sheaf on  $X$ . The boundary map

$$H^1(D, \mathcal{K}_1^D) = H^1(X, j_* \mathcal{K}_1^D) \longrightarrow H^2(X, \mathcal{K}_2)$$

is the map  $CH^1(D) \rightarrow CH^2(X)$ . For any divisor  $\alpha$  of  $D$ , the  $\mathcal{K}_1^D$ -torsor  $\mathcal{O}_\alpha$  determines a unique  $j_* \mathcal{K}_1^D$ -torsor  $L_\alpha$  on  $X$ . The  $K_2$ -gerbe  $\mathcal{C}_\alpha$  (viewing  $\alpha$  as a codimension two cycle on  $X$ ) is the lifting gerbe of the  $j_* \mathcal{K}_1^D$ -torsor  $L_\alpha$  (obstructions to lifting to a  $\mathcal{F}_2$ -torsor).

This generalizes to higher codimensions (and pursued in forthcoming work):

- (codimension three) If  $\beta$  is a codimension-two cycle of  $D$ , then the gerbe  $\mathcal{C}_\beta$  on  $D$  determines a unique gerbe  $L_\beta$  on  $X$  (with band  $j_* \mathcal{K}_2^D$ ). The obstructions to lifting  $L_\beta$  to a  $\mathcal{F}_3$ -gerbe is a 2-gerbe  $\mathcal{G}_\beta$  with band  $\mathcal{K}_3$  on  $X$ . This gives an example of a higher gerbe invariant of a codimension three cycle on  $X$ . Gerbes with band  $K_3(F_X)/\mathcal{K}_3$  provide the codimension-three analog of Cartier divisors  $H^0(X, K_1(F_X)/\mathcal{K}_1)$ .

- (Parshin chains) Recall that a chain of subvarieties

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots \hookrightarrow X_n = X$$

where each  $X_i$  is a divisor of  $X_{i+1}$  gives rise to a Parshin chain on  $X$ . We will call a Parshin chain smooth if all the subvarieties  $X_i$  are smooth. Iterating the previous construction provides a higher gerbe on  $X_n = X$  with band  $\mathcal{K}_j$  attached to  $X_{n-j}$  (a codimension  $j$  cycle of  $X_n$ ).

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